(1) Let \( f : \mathbb{A}^1 \to \mathbb{A}^1 \) be a finite map.
   (a) Prove that \( y \in \mathbb{A}^1 \) is a ramification point iff the graph of \( f \) has an intersection with multiplicity \( m > 1 \) with the fiber of the second projection \( \mathbb{A}^1 \times \{y\} \).
   (b) Show that if the base field \( k \) has characteristic zero then \( f \) has a ramification point unless \( f \) is an isomorphism.
   [In fact, this is true more generally for a finite morphism from an irreducible curve to \( \mathbb{A}^1 \)].
   (c) Show that the Artin-Schreier map \( f(x) = x^p - x, p = \text{char}(k) \) has no ramification points.

(2) For an algebraic variety \( X \) over a field \( k \) of characteristic \( p \) the Frobenius twist \( X' \) of \( X \) is defined as follows.
\[ X' = X \text{ as a topological space. A function } f \text{ on } U' \subseteq X' \text{ is regular iff } f(x) = \phi(x)^p \text{ where } \phi \text{ is a regular function on } U = U' \subseteq X. \]
\[ \text{The identity map } X \to X' \text{ defines a morphism } Fr : X \to X' \text{ called the Frobenius morphism.} \]
[Notice though that it does not define a morphism from \( X' \) to \( X \).]
   (a) Check that if \( X \) is a closed subvariety in \( \mathbb{A}^n \) or \( \mathbb{P}^n \) whose ideal is generated by polynomials with coefficients in \( \mathbb{F}_p \), then \( X' \cong X \). Moreover, we have an isomorphism such that that composition \( X \xrightarrow{Fr} X' \cong X \) is given by \( (x_i) \mapsto (x_i^p) \).
   (b) Let \( X \) be a normal irreducible variety of dimension \( n \). Prove that \( Fr : X \to X' \) is finite, find its degree and prove that every point is its ramification point.
   [Hint: reduce to the case of \( X = \mathbb{A}^n \)].
   (c) Describe the intersection points of the graph of Frobenius \( Fr : \mathbb{A}^1 \to \mathbb{A}^1 \) with the diagonal and check that each one has multiplicity one.

(3) Let \( X \) be the line with a double point at zero, thus we have a map \( X \to \mathbb{A}^1 \) which is an isomorphism over \( \mathbb{A}^1 \setminus \{0\} \) and the preimage of 0 consists of two points.
   (a) Let \( Y = \mathbb{A}^2 \setminus \{0\} \). Show that the map \( m : Y \to \mathbb{A}^1, m(x,y) = xy \) can be lifted to an onto map \( Y \to X \); moreover, there are two distinct such liftings.
   (b) Describe the closure of the diagonal in \( X^2 \) and \( X^3 \). More precisely, define a map from that closure to \( \mathbb{A}^1 \), which is an isomorphism over \( \mathbb{A}^1 \setminus \{0\} \) and count the number of points in the preimage of zero.

(4) \( X \subseteq \mathbb{A}^{n+1} \) is the zero set of a polynomial \( P \) which is irreducible and has the form \( P = P_d + P_{d+1} \) where \( P_d, P_{d+1} \) are nonzero homogeneous polynomials of degrees \( d, d+1 \) respectively. Prove that \( X \) is birationally equivalent to \( \mathbb{A}^n \).