Lecture 2: Affine Varieties

Side Remark Recall that we introduced three types of questions in the last lecture: counting over \( \mathbb{C} \), counting over \( \mathbb{F}_q \), and the slope of the set of solutions over \( \mathbb{C} \). It is worth pointing out that there is indeed a connection between the two latter types, as sketched out by the Weil conjectures.

Last time we defined Spec \( A \), where \( A \) is a finitely generated \( k \)-algebra with no nilpotents. Namely, Spec \( A = \text{Hom}_{k-\text{alg}}(A,k) = \{ \text{maximal ideals in } A \} \). Zariski closed set are defined in [Kem93]. Now recall that there is a bijection between Zariski closed subsets of Spec \( A \) and the radical ideals of \( A \). Suppose \( Z_1, Z_2 \) correspond to \( I_1, I_2 \), then \( Z_1 \cup Z_2 \) corresponds to \( I_1 \cap I_2 \). Note that \( I_1 + I_2 \) may not be reduced even if \( Z_1, Z_2 \) are varieties. For instance, let \( A = k[x,y] \), \( I_1 = (y-x^2) \), \( I_2 = (y) \), then \( A/(I_1 + I_2) = k[x]/x^2 \).

Theorem 1.1. Let \( k[U] \) denote functions associated with the set \( U \), as specified in last lecture. Then \( k[\text{Spec } A] \cong A \).

Proof. (This was done in [Kem93], Section 1.3-1.5.) Recall that as a set, Spec \( A \) is \( k - \text{Hom}(A,k) \), because each maximal ideal is the kernel of a homomorphism \( A \to k \) and vice versa. So there’s a map \( \phi : A \to k[\text{Spec } A] \) given by \( a \mapsto (x \mapsto x(a)) \), which we shall prove to be a bijection.

We first want the topological structure on Spec \( A \). This is given by \( Z(I) = \{ x \in \text{Spec } A \mid i(x) = 0 \forall i \in I \} \), where \( I \) is a subset of \( A \). One can directly check that this gives a topology on Spec \( A \). Next we need to make it a space with functions. The construction is given as: \( k[U] = \{ f : U \to k \mid \exists (U_\alpha, a_\alpha, b_\alpha), \bigcup \alpha U_\alpha = U, f|_{U_\alpha} = \phi(a_\alpha)/\phi(b_\alpha), \phi(b_\alpha)(x) \neq 0 \forall x \in U_\alpha \} \).

To show injectivity, let \( a \neq 0 \in A \), then we need to find some \( x : A \to k \) in Spec \( A \) such that \( \phi(a)(x) = x(a) \neq 0 \). To do so we’d need the following fact, the proof of which is standard commutative algebra:

Lemma 1 (Noether Normalization). Given \( A \) a finitely-generated \( k \)-algebra, there exists some algebraically independent elements \( X_1, \ldots, X_d \) over \( k \) such that \( A \) is a finitely generated \( k[X_1, \ldots, X_d] \)-module.

Apply this fact with the localization \( A_0 \), which is nonempty because \( A \) has no nilpotent (otherwise if \( 1 = 0 \) in the localization ring, then \( a^n = a^n \cdot 1 = 0 \), and is finitely generated as we just need to add \( 1/a \) to \( A \). Thus we get some \( X_1, \ldots, X_d \) such that \( A_0 \supseteq B = k[X_1, \ldots, X_d] \), then there is a surjection \( \psi : k - \text{Hom}(A_0, k) \to k - \text{Hom}(B, k) \). Let \( \varphi \neq 0 \in k - \text{Hom}(B, k) \), and let \( \psi(\varphi) = \varphi \), and let \( x = A \to A_0 \).

Now we need surjectivity. Take \( f \in k[\text{Spec } A] \) and we need to show it is in \( A \). Assume the data is given by \( (U_\alpha, a_\alpha, b_\alpha) \), where we can assume that each \( U_\alpha = D(c_\alpha) \). By the replacement \( a_\alpha \to a_\alpha c_\alpha \), \( b_\alpha \to b_\alpha c_\alpha \), one can assume that \( U_\alpha = D(b_\alpha) \). Since the \( D(b_\alpha) \) sets cover Spec \( A \), we know that the ideal generated by \( \{ b_\alpha \} \) corresponds to empty set, thus by Nullstellensatz (c.f. [Kem93], Theorem 1.4.5), there must be some finite set \( b_1, \ldots, b_m \) and some constants \( z_1, \ldots, z_m \in A \) such that \( \sum_{i=1}^m z_i b_i^2 = 1 \in A \). Now \( b_\alpha f \) agrees with \( a_\alpha b_\alpha \) both on \( U_\alpha \) and its complement, so they are equal in \( A \), which means \( f = f \cdot 1 = \sum_i z_i (f b_i^2) = \sum_i z_i a_i b_i \in A \).

Note this last part can also give us the following:

Proposition 1. Spec \( A \) is quasi-compact for any commutative ring \( A \).

Proof. Take a covering \( X = \bigcup U_\alpha \), then can pick \( U_{f_\alpha} \subseteq U_\alpha \), then we have \( (f_\alpha) = 1 \), and thus there’s a finite subset \( (f_{d_1}, \ldots, f_{d_n}) = 1 \).

What we really want to say is:

Theorem 1.2. Given a space of functions \( X \), \( X \) is an affine variety if and only if \( X = \text{Spec } A \) for a finitely generated commutative ring \( A \) with no nilpotents.

Proof. Let’s show that Spec \( A \) is affine; the other direction will be done in the next lecture. Let \( X \) be any space with functions, then we need to show that \( \ast : \text{Morphism}(X, \text{Spec } A) \to k - \text{Hom}(A,k[X]) \) is injective and surjective. For injectivity, let \( f : X \to \text{Spec } A \) be a morphism and let \( x \) be any point on \( X \), then \( \delta_{f(x)} \),
the evaluation map at \( f(x) \), is given by \( \delta_f(x) = a(f(x)) = (f^*a)(x) \) for \( a \in A \), i.e. \( f(x) \), equivalently \( \delta_f(x) \), is specified by \( x \) and \( f^* \). On the other hand, define \( \ast^{-1} \) by \( \delta_{\ast^{-1}}(g)(x) = \delta_x \circ g \), then one can check this gives a well-defined inverse to \( \ast \) and thus \( \ast \) is bijective.

**Definition 1.** An algebraic variety over \( k \) is a space with functions which is a finite union of open subspaces, each one is an affine variety.

**Lemma 2.** A closed subspace in an affine variety is also affine, and global regular functions restrict surjectively.

**Proof.** \( X = \text{Spec} \ A, Z = Z_I, I \) is a radical. Then \( Z_I \cong \text{Spec}(A/I) \). Surjectivity follows from the fact that \( k[\text{Spec} \ A] = A \). \( \square \)

**Corollary 1.** A closed subspace of a variety is a variety.

**Theorem 1.3 (Hilbert Basis Theorem).** \( k[x_1, \ldots, x_n] \), and hence any finitely generated \( k \)-algebra is Noetherian.

**Corollary 2.** An algebraic variety is a Noetherian topological space (that is, every descending chains of closed subsets terminate; equivalently, every open subset is quasicompact).

**Corollary 3.** An open subspace of an algebraic variety is an algebraic variety. (Contrast with affine variety.)

**Proof.** Need to check that an open subset of an affine variety is covered by finitely many affine varieties. This follow from quasi-compactness. \( \square \)

Combine the two corollaries above, we see that a locally closed subspace (intersection of open and closed) of an algebraic variety is again a variety. However, the union of an open set and a closed set need not be a variety. For an counterexample, consider \( (A^2 - \{ x = 0 \}) \cup \{ 0 \} \).

**Definition 2 (Projective Space).** Topologically, the projective space \( \mathbb{P}^n \) is given by the quotient topology \( \mathbb{A}^{n+1} - \{ 0 \} / (x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n) \forall \lambda \neq 0 \). A function on \( U \subseteq \mathbb{P}^n \) is regular if its pullback by \( \mathbb{A}^{n+1} - \{ 0 \} \xrightarrow{\pi} \mathbb{P}^n \) is regular on \( \pi^{-1}(U) \).

**References**
