Lecture 6: Function Field, Dominant Maps

Definition 1. Let \( X \) be an irreducible variety. The function field of \( X \), denoted \( k(X) \) is defined as the limit

\[
K(X) = \lim_{U \subseteq X} k[U]
\]

taken over all open subsets of \( X \) with the obvious restriction morphisms.

If \( X \) is irreducible, \( k(X) \) is just the fraction field of the integral domain \( k[U] \) for any open affine subset \( U \subseteq X \). A morphism of varieties \( f : X \to Y \) is dominant if the image of \( f \) is dense. Suppose \( f : X \to Y \) is dominant and \( \phi \) is a rational function on \( Y \). Then by definition \( \phi \) is an equivalence class \( (U, g \in k[U]) \), where \( (U, g) \) and \( (U', g') \) are equivalent if they restrict to the same function on an open subset of \( U \cap U' \). Pick a representative \( (U, g) \) for \( \phi \). Since \( f(X) \) is dense, \( f^{-1}(U) \) is non-empty. Hence, \( (f^{-1}(U), f^*g) \) is a rational function on \( X \). It is easy to see that ‘equivalent’ functions on \( Y \) pull back to ‘equivalent’ functions on \( X \). Thus, we obtain a map of function fields \( f^* : k(Y) \to k(X) \).

Definition 2. For any dominant map of irreducible varieties \( f : X \to Y \) we obtain a field extension \( k(X)/f^*k(Y) \). The degree of \( f \) is the degree of this field extension.

Lemma 1. Let \( X \) and \( Y \) be irreducible varieties with \( Y \) normal and \( f : X \to Y \) a finite dominant map. Then for any \( y \in Y \), \( \#f^{-1}(y) \leq \deg f \).

Proof. Since \( f \) is finite (hence affine) we may reduce to the case where \( X = \text{Spec}(A) \) and \( Y = \text{Spec}(B) \). Finiteness implies that \( A \) is a finitely-generated \( B \)-module. Suppose \( \#f^{-1}(y) = m \) and let \( \phi \in A \) be a function taking distinct values on the elements of \( f^{-1}(y) \). Let \( P \in B[t] \) be the minimal polynomial for \( \phi \). Then \( \deg(P) \leq \deg(f) \). Since \( Y \) is normal, \( B \) is integrally closed. Hence, the coefficients of \( P \) are elements of \( B \) and are therefore constant on \( f^{-1}(y) \). Let \( \hat{P} \in k[t] \) denote the polynomial obtained from \( P \) by replacing the coefficients with their values at \( y \). \( \hat{P} \) has at least \( m \) roots and hence \( m \leq \deg(\hat{P}) = \deg(P) \leq n \) which completes the proof.

Definition 3. Let \( X, Y \) be irreducible varieties, and let \( f : X \to Y \) be a dominant map of degree \( n \). \( f \) is unramified over \( y \in Y \) if \( \#f^{-1}(y) = n \). Otherwise, we say that \( f \) is ramified at \( y \) or that \( y \) is a ramification point of \( f \).

Proposition 1. Let \( f : X \to Y \) be a finite dominant map of irreducible varieties and let \( R \subseteq X \) be the set of ramification points. \( R \) is a closed subset of \( X \) and if the field extension \( k(X)/f^*k(Y) \) is separable, then \( R \neq X \).

Proof. Since \( f \) is finite (hence affine), we may reduce to the case where \( X, Y \) are affine. We will first prove that \( Y - R \) is open. Suppose \( f \) is unramified over \( y \). Choose \( \phi \) as in the proof of lemma 1. Since \( f \) is unramified at \( y \), \( \phi \) has \( n \) distinct roots, where \( n = \deg(f) \). Write \( D(\phi) \) for the discriminant of \( f \). \( D(\phi) = D(\phi)(y) \neq 0 \) implies \( f \) unramified at \( y \). But \( D(\phi)(y') \neq 0 \) for \( y' \in \) a neighborhood of \( y \) by continuity. Hence, \( Y - R \) is open. Suppose \( k(X)/f^*k(Y) \) is separable. Then \( k(X) \) is generated over \( f^*k(Y) \) by a single element \( a \in A \) by field theory. Let \( F \) denote the minimal polynomial for \( a \). Then \( \deg(F) = n \) and \( D(F) \neq 0 \) since \( F \) has no repeated roots. Hence, there are elements \( y \in Y \) with \( D(F)(y) \neq 0 \). These will not be ramification points of \( f \).

We finish the lecture by stating an easy but extremely important general categorical result called Yoneda’s Lemma. It says roughly that an object in a category is uniquely determined by a functor it represents. The standard way to apply it in algebraic geometry is as follows. Due to Yoneda’s Lemma, to define an algebraic variety \( X \), it suffices to describe the functor represented by \( X \) and then check that the functor is representable. This a standard tool used to make sense of the intuitive idea "the variety \( X \) parametrizing algebraic (or algebro-geometric) data of a given kind" — such as the Grassmannian variety parametrizing linear subspaces of a given dimension in \( k^n \). More complicated examples (beyond the scope of 18.725) involve subvarieties in a given variety with fixed numerical invariants etc. In the next lecture we will use Yoneda Lemma to define products of algebraic varieties.
Lemma 2 (Yoneda). Let $C$ be a category. For every $x \in C$ define a covariant functor

$$h^x : C \to \text{Set}$$
$$c \mapsto \text{Hom}(x, c)$$

Then the assignment $x \mapsto h^x$ defines a functor $h : C \to \text{Functors}(C, \text{Set})$. $h$ is fully faithful and therefore injective on objects (up to isomorphism).