Today we will prove the uniqueness of minimal models of non-ruled surfaces (in characteristic 0) and talk about the characterization of ruled surfaces.

**Theorem 1** (Grothendieck-Cartier). In characteristic 0, a group scheme $G$ is always reduced.

**Proposition 1.** Let $X$ be a surface, $\alpha : X \to \text{Alb}(X)$ the Albanese map. Suppose $\alpha(X)$ is a curve $C$. Then $C$ is a smooth curve of genus $q$, and the fibers of $\alpha$ are connected.

**Lemma 1.** Suppose $\alpha$ factors as $X \xrightarrow{f} T \xrightarrow{j} \text{Alb}(X)$ with $f$ surjective. Then $j : \text{Alb}(T) \to \text{Alb}(X)$ is an isomorphism.

**Lemma 2.** Let $X$ be a surface with $p_g = 0, q \geq 1, \alpha : X \to \text{Alb}(X)$ its Albanese map. Then $\alpha(X)$ is a curve.

**Proof.** If $Y = \alpha(X)$ is a surface, then the morphism $\alpha' : X \to Y$ is generically finite, hence generically étale (in characteristic 0). Pick a smooth point $y \in Y$, and find an invariant differential form $\omega : H^0(\text{Alb}(X), \Omega^2)$ which is nonzero at $y$ (since $\text{Alb}(X)$ is an abelian variety). Then $\alpha^*\omega$ is a nonzero element of $H^0(X, \omega_X)$, contradicting $p_g = 0$.

**Theorem 2.** Let $X, X'$ be two nonruled minimal surfaces. Then every birational map from $X$ to $X'$ is an isomorphism. In particular, every nonruled surface admits a unique minimal model up to isomorphism. The group of birational maps from a nonruled minimal surface to itself coincides with the group of automorphisms of the surface.

**Proof.** (In characteristic 0: holds in positive characteristic with some modifications.) Let $\phi : X' \dasharrow X$ be a birational map. Then $\exists$ a series of blowups $\pi_1 \circ \cdots \circ \pi_n : \tilde{X} \to X$ resolving $\phi$ to a morphism $f : \tilde{X} \to X$. Choose one with $n$ minimal. If $n = 0$, we are done, so assume that $n \geq 1$. Let $E$ be the exceptional curve of the blowup $\pi_n$. Then $f(E)$ is a curve in $X$, otherwise $f$ would factor as $f' \circ \pi_n$ contradicting minimality of $n$. Now, calculate $C \cdot K_X$. If $\pi : \tilde{Y} \to Y$ is a blowup of a point $p$ on a surface $Y$, and $\tilde{D}$ is an irreducible curve in $\tilde{Y}$ such that $\pi(\tilde{D})$ is a curve $D$, then we have $K_{\tilde{Y}} \cdot \tilde{D} = K_Y \cdot D + m \geq K_Y \cdot D$, where $m$ is the
multiplicity of $D$ at $p$, i.e. $E \cdot \tilde{D}$. Equality holds iff $\tilde{D}$ doesn’t intersect the exceptional divisor. Since $f$ is composed of blowups, we get $K_X \cdot C \leq K_X \cdot E = -1$ with equality iff $E$ doesn’t meet any curve contracted by $f$. But in that case, $f$ restricted to $E$ is an isomorphism, so $C$ is a rational curve with $K \cdot C = -1$, contradicting the minimality of $X$. So $K_X \cdot C \leq -2$, and $C^2 \geq 0$ by the genus formula. Now, this implies that all the plurigenera vanish, for if $|nK|$ contained an effective divisor $D$ for $n \geq 1$, then $D \cdot C \geq 0$ by the useful lemma and $K_X \cdot C \geq 0$, a contradiction. If $q = 0$, Castelnuovo’s theorem (for $q = 0, p_2 = 0$) implies that $X$ is rational, excluded by hypothesis. If $q > 0$, $X \rightarrow \text{Alb}(X)$ gives a surjective morphism $p : X \rightarrow B$ with connected fibers, where $B$ is a smooth curve of genus $q > 0$. Since $C$ is rational, $C$ is contained in a fiber of $p$, and since $C^2 \geq 0$, we must have $F = rC$ for some $r$, so $C^2 = 0 \implies C \cdot K = -2$. Again, the genus formula gives $r = 1, g(F) = 0$ and $C$ smooth, which by Noether-Enriques implies that $X$ is ruled, with is also excluded.

We now go on to separate surfaces into the following types.

(a) There is an integral curve $C$ on $X$ with $K \cdot C < 0$.
(b) For every integral curve $C$ on $K$, we have $K \cdot C = 0$, i.e. $K \equiv 0$.
(c) $K^2 = 0, K \cdot C \geq 0$ for every integral curve $C$ on $X$, and there is at least one integral curve $C'$ s.t. $K \cdot C' > 0$.
(d) $K^2 > 0$, and $K \cdot C \geq 0$ for every integral curve $C$ on $X$.

We will show that:

1. $X$ is in class (a) $\iff \kappa(X) = -\infty \iff p_4 = p_6 = 0 \iff p_{12} = 0$
2. $X$ is in class (b) $\iff \kappa(X) = 0 \iff 4K \sim 0$ or $6K \sim 0 \iff 12K \sim 0$.
3. $X$ is in class (c) $\iff \kappa(X) = 1 \iff |4K|$ or $|6K|$ has a strictly positive divisor at $K^2 \iff |12K|$ has a strictly positive divisor and $K^2 = 0$.
4. $X$ is in class (d) $\iff \kappa(X) = 2 \iff |2K| \neq \emptyset$.

Proof. We demonstrate this following Mumford, Mumford-Bombieri, and Badescu. First, let us see that every surface is exactly in one of the classes above. Mutual exclusivity is obvious. If $X$ is not in any of the four classes, then $K^2 < 0$ and $K \cdot C \geq 0$ for every curve $C$ on $X$. We can exclude this case as follows: let $H$ be a hyperplane section, $D = aK + bH$ for $a,b$ natural numbers. Then $D^2 = a^2K^2 + 2abK \cdot H + b^2H^2 = a^2P(b/a)$ for $P(t) = H^2t^2 + 2(K \cdot H)t + K^2$. By our hypothesis, $P$ is an increasing function on $[0, \infty)$, is eventually positive, and $P(0) < 0$, implying that it has a unique root $t_0$. For $b/a > t_0, 0 < a^2P(b/a) = D^2$. Also, for every integral $C, D \cdot C = a(K \cdot C) + b(H \cdot C) > 0$. By Nakai-Moishezon, such a $D$ is ample, so $nD$ is very ample for $n >> 0$ and $K \cdot D \geq 0$. Thus, for $t > t_0, (K \cdot H)t + K^2 \geq 0$, and by continuity, the same is true for $t = t_0$. $P(t_0) \geq H^2t_0^2 - K^2 > 0$, giving us a contradiction.
Now we begin to prove or equivalences. To show (i), we need to show that (a) \[X\] is ruled. In fact, we can replace (a) by saying that \(\exists\) an effective divisor \(D\) on \(X\) s.t. \(K \cdot D < 0\).

- Step 1: there is an ample \(H\) s.t. \(K \cdot H < 0\). To see this, note that if \(C^2 < 0\), then \(K \cdot C + C^2 = 2p_a(C) - 2 \geq -2\), implying that \(K \cdot C = C^2 = -1\) and so \(X\) is not minimal. Thus, \(C^2 \geq 0\). Let \(H_1\) be an ample divisor on \(X\). Then, for all \(n \geq 0\), \(nC + H_1\) is ample by Nakai-Moishezon, and for \(n >> 0\), \(K \cdot (nC + H_1) < 0\) so we’re done.

- Step 2: If \(K^2 > 0\), then \(X\) is rational, hence ruled. Noether’s formula gives \(12\chi(\mathcal{O}_X) = K^2 + c_2 = K^2 + 2 - 2b_1 + b_2\). Since \(p_g = 0\) (if \(|nK|\) were effective, \(nK \cdot H\) would be positive, contradicting Step 1), it follows that the Picard scheme is reduced, \(b_1 = 2q\), and \(10 = 8q + K^2 + b_2\). If \(K^2 > 0\), then \(q = 0\) or \(1\) is forced. If \(q = 1\), then since \(q = s = \dim \text{ Alb} (X)\), there is a morphism \(X \to E\) to an elliptic curve, and so \(b_2 \geq 2\) (Pic has the class of a fiber and class of a hyperplane section). This is impossible, so \(q = 0\). By Castelnuovo, \(X\) is rational and thus ruled.

- Step 3: If \(K^2 \leq 0\), then for all \(n\), there is an effective divisor \(D\) on \(X\) s.t. \(|D + K| = \emptyset\) and \(\dim |D| \geq n\). To see this, Let \(H\) be an ample divisor s.t. \(K \cdot H < 0\). For all \(n\), \((nH + mK) \cdot H < 0\) for \(m >> 0\) (depending on \(n\)), so \(nH + mK\) can’t be linearly equivalent to an effective divisor for \(m >> 0\). Let \(m_n\) be a nonnegative integer s.t. \(|nH + m_nK| \neq \emptyset\) but \(|nH + (m_n + 1)K| = \emptyset\). Let \(D_n \in |nH + m_nK|\), and write it as \(D'_n + D''_n\), where each summand is positive and the components \(E\) of \(D'_n\) satisfy \(E \cdot K < 0\), while those of \(D''_n\) satisfy \(E \cdot K \geq 0\). Note that \(E \cdot K < 0 \implies E^2 \geq 0\) (\(E\) not exceptional), so \((D'_n)^2 \geq 0\). Next, \(|K - D'_n| \subset |K| = \emptyset\), so by Serre duality \(H^2(\mathcal{O}_X(D'_n)) = 0\). Riemann-Roch gives that

\[
\dim |D'_n| = h^0(\mathcal{O}_X(D'_n)) - 1 \geq \chi(\mathcal{O}_X(D'_n)) - 1
\]
\[
\geq \frac{(D'_n \cdot (D'_n - K))}{2} + \chi(\mathcal{O}_X) - 1
\]
\[
\geq \frac{-D'_n \cdot K}{2} + \chi(\mathcal{O}_X) - 1
\]
\[
\geq \frac{-D_n \cdot K}{2} + \chi(\mathcal{O}_X) - 1
\]
\[
\geq \frac{-n(H \cdot K)}{2} - \frac{m_nK^2}{2} + \chi(\mathcal{O}_X) - 1
\]
\[
\geq \frac{n}{2} + \chi(\mathcal{O}_X) - 1 \to \infty \text{ as } n \to \infty
\]

(1)

Also, \(|K + D'_n| \subset |K + D_n| = |nH + (m_n + 1)K| = \emptyset|.
• Step 4: If $D$ is an effective divisor s.t. $|K + D| = \emptyset$, then the natural map $\text{Pic}^0(X) \to \text{Pic}^0(D)$ is surjective. To see this, note that $h^0(\mathcal{O}_X(K + D)) = h^2(\mathcal{O}_X(-D)) = 0$. Now, $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ gives that $H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_D)$ is surjective. These are the tangent spaces at 0 to the connected and reduced group schemes $\text{Pic}^0(X)$ and $\text{Pic}^0(D)$ ($\text{Pic}^0(X)$ is reduced since $p_g = 0$ so $\Delta = 0$). Thus, the desired map is surjective.

• Step 5: If $D$ is an effective divisor s.t. $|K + D| = \emptyset$ and if $D = \sum n_i E_i$, then

1. All the $E_i$ are nonsingular, and $\sum p_a(E_i) \leq q = h^1(X, \mathcal{O}_X)$.
2. $\{E_i\}$ is a configuration of curves with no loops, and $E_i$ intersect transversely.
3. If $n_i \geq 2$ then either
   (a) $E_i$ is rational,
   (b) $(E_i)^2 < 0$, or
   (c) $E_i$ is an elliptic curve with $E_i^2 = 0$ and the normal bundle of $E_i$ in $X$ is nontrivial.

$\square$