1. **Elliptic and Quasi-elliptic Surfaces**

1.1. **Preliminary theorems.**

**Theorem 1.** Let \( f : X \to Y \) be a dominant morphism from an irreducible, smooth algebraic variety \( X \) to an algebraic variety \( Y \) s.t. \( f^* : k(Y) \to k(X) \) is separable and \( k(Y) \) is algebraically closed in \( k(X) \). Then \( \exists \) a nonempty open subset \( V \subset Y \) s.t. \( \forall y \in V, \) the fiber \( f^{-1}(y) \) is geometrically integral.

*Sketch of proof.* \( k(Y) \to k(X) \) separable, \( k(Y) \) algebraically closed in \( k(X) \), implies that the generic fiber is geometrically integral over \( k(Y) \) (see Milne’s Algebraic Geometry for instance). Then some geometric reasoning shows that \( \exists \) an open set for which this is true (i.e. the set for which \( f^{-1}(y) \) is not geometrically integral is a constructible set). For full proof, see Badescu, Algebraic Surfaces, p. 87-90.

Let \( K \) be an algebraic function field in one variable over a perfect field \( k \) (i.e. \( K \) is a finitely generated extension of \( k \) of transcendence degree 1) and \( L \supset K \) is an extension of \( K \) s.t. \( K \) is algebraically closed in \( L \).

**Theorem 2.** \( L/K \) is separable.

**Corollary 1.** Let \( f : X \to Y \) be a dominant morphism from an irreducible smooth variety \( X \) of dimension \( \geq 2 \) to an irreducible curve \( Y \) s.t. \( k(Y) \) is algebraically closed in \( k(X) \). Then the fiber \( f^{-1}(y) \) is geometrically integral for all but finitely many closed points \( y \in Y \).

**Theorem 3.** Let \( f : X \to Y \) be a dominant morphism of smooth irreducible varieties over an algebraically closed field of characteristic 0. Then \( \exists \) a nonempty open \( V \subset Y \) s.t. \( f|_{f^{-1}(V)} : f^{-1}(V) \to V \) is a smooth morphism, i.e. for every \( y \in V \), the fiber \( f^{-1}(y) \) is geometrically smooth of dimension \( \dim X - \dim Y \).

**Definition 1.** A surjective morphism \( f : X \to B \) from a surface \( X \) to a smooth projective curve \( B \) is called a fibration.

*Remark.* \( f \) is necessarily flat (since it is surjective and \( B \) is a curve). Thus, the arithmetic genus of the fibers \( F_b \) for \( b \in B \) is constant.
Definition 2. $f$ is called a (relatively) minimal fibration if $\forall b \in B, F_b$ does not contain any exceptional curves of the first kind.

Let $f : X \to B$ be a fibration s.t. $f_*\mathcal{O}_X = \mathcal{O}_B$. By Zariski’s main theorem, the fibers of $f$ are connected. The condition is equivalent (by the second corollary) to $k(B)$ being algebraically closed in $k(X)$. So all but finitely many fibers are integral curves.

Definition 3. A fibration $f$ is elliptic if $f$ is minimal, $f_*\mathcal{O}_X = \mathcal{O}_B$, and almost all the fibers of $f$ are smooth curves of genus 1 (i.e. elliptic curves). If the fibers are singular integral curves of arithmetic genus 1, $f$ is called quasi-elliptic.

Note. Suppose $f : X \to B$ is quasi-elliptic. If any fiber of $f$ is smooth, then the morphism $f$ is generically smooth, i.e. $f$ is elliptic. So for $f$ to be quasi-elliptic, all the fibers must be singular. By the third theorem, quasi-elliptic fibrations cannot exist in characteristic 0.

Proposition 1. Quasi-elliptic fibrations only exist in characteristics 2 and 3. The general fiber of $f$ is a rational projective curve with one singular point, an ordinary cusp.

Proof. Let $b \in B$ be a closed point s.t. $F_b$ is integral. Then $p_a(F_b) = 1$, $F_b$ singular $\implies$ $F_b$ is a rational curve with exactly one singular point which is a node or a cusp. Let’s see that we can’t have nodes. Let $\Sigma$ be the set of points $x \in X$ where $f$ is not smooth. Remove the (finite number of) points $b_1, \ldots, b_n \in B$ s.t. $F_{b_i}$ is not integral, and set $\Sigma_0 = \Sigma \cap f^{-1}(B \setminus \{b_1, \ldots, b_n\})$.

Choose $x \in \Sigma_0$, and let $b = f(x)$. Let $t$ be a regular local parameter at $b$ and $u, v$ regular local parameters at $x \in X$. Then $f$ is given locally by $f(u, v) \in k[[u, v]]$, a formal power series corresponding to the completed local homomorphism $k[[t]] = \widehat{\mathcal{O}_{B, b}} \to \widehat{\mathcal{O}_{X, x}} = k[[u, v]]$. Since $F_b$ is integral, we can choose $u, v$ s.t. $f(u, v)$ has the form

1. $f(u, v) = G(u, v)(u^2 + v^3)$ (cusp)
2. $f(u, v) = G(u, v)uv$ (node)

with $G(0, 0) \neq 0$. We have in case (2)

$$(1) \quad \frac{\partial f}{\partial u} = G(u, v)v + \frac{\partial G}{\partial u}(u, v)uv = v(\text{unit})$$

(since $G(0, 0) \neq 0$) and similarly $\frac{\partial f}{\partial v} = u(\text{unit})$. So $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial v} = 0$ cuts out the single point 0, 0 and $x$ is an isolated point of $\Sigma_0$. But then $f$ is smooth away from $x$ in a neighborhood of $x$, so by properness $\exists$ an open set $V \subset B \setminus \{b_1\}$ containing $B$ s.t. the restriction of $f$ to $f^{-1}(V)$ is smooth, implying that $f$ is not quasi-elliptic. Thus, $f$ must have a cusp everywhere away from $F_b$. Then $\Sigma_0$ is the locus of all such cuspidal points on $f^{-1}(B \setminus \{b_1\})$ and $f(u, v) = G(u, v)(u^2 + v^3), G(0, 0) = 0$. 
If \( \text{char} \ k \neq 2 \), then

(2) \[
\frac{\partial f}{\partial u} = u \left[ 2G(u, v) + v^3 \frac{\partial G}{\partial u} (u, v) \right] = 0
\]
defines, near \( x \), a smooth curve that contains \( \Sigma_0 \). Thus, \( \frac{\partial f}{\partial u} \) is a local equation of \( \Sigma_0 \) at \( x \) and \( \Sigma_0 \) is smooth at each of its points. The restriction of \( f \) to \( \Sigma_0 \) is a bijection of \( \Sigma_0 \) onto \( B \setminus \{b_i\} \). The intersection number \( \Sigma_0 \cdot F_b, b \in B \setminus \{b_i\} \) is

(3) \[
\Sigma_0 \setminus F_b \lambda = \dim (\mathcal{O}_{\Sigma_0,x}/\mathfrak{m}_{B,b}\mathcal{O}_{\Sigma_0,x}) = \dim k[[u, v]]/(f, \frac{\partial f}{\partial u})
\]

\[
= \dim k[[u, v]]/(u^2 + v^3, u) = \dim k[v]/v^3 = 3
\]
The field extension \( k(B) \hookrightarrow k(\Sigma_0) \) is finite, purely inseparable (since \( \Sigma_0 \) is irreducible) and so it has degree \( p^m \) for some \( m \geq 0 \) \( (p = \text{char} \ k) \). Thus, \( p^m = \Sigma_0 \cdot F_b = 3 \) for any \( b \in B \setminus \{b_i\} \), implying that \( \text{char} \ k = 3 \). \( \Box \)

2. Elliptic surfaces

Let \( f : X \to B \) have generic fiber a smooth elliptic curve.

**Theorem 4.** If \( f \) is smooth, \( \exists \) an étale cover \( B' \) of \( B \) s.t. \( f' : X' = X \times_B B' \to B \) is trivial \( (i.e. \) is a product \( B' \times F) \).

**Sketch of proof.** We have a \( J \)-map \( B \to \mathbb{A}^1 \) given by \( b \mapsto j(F_b) \). Since \( B \) is complete, any map to \( \mathbb{A}^1 \) is constant, so the \( j \)-invariant is constant on fibers. Now eliminate automorphisms by rigidifying (i.e. use full level \( N \) structure, e.g. with \( N \geq 4 \)). \( \Box \)

Now, we’ll consider the case where we have at least one singular fiber. Given \( f : X \to B \), Tate’s algorithm computes the singular fibers of the Néron model (minimal proper regular model). Recall the Weierstrass equation for an elliptic curve:

(4) \[
y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6
\]
If we consider an elliptic surface, \( a_i \in k(B) \), we can work locally at \( b \in B \) and use Tate’s algorithm. For \( B = \mathbb{P}^1, a_i = a_i(t) \in k(t) \). We can clear denominators by multiplying \( x, y \) by \( \lambda^2, \lambda^3 \) to make \( a_i \in k[t] \). We find that the singular fibers \( C_p \) fall into the following Kodaira-Néron classification:

- **Type I \( \mathbf{I}_1 \)** \( C_p \) is a nodal rational curve.
- **Type I \( \mathbf{I}_n \)** \( C_p \) consists of \( n \) smooth rational curves meeting with dual graph \( \widetilde{A}_n \), i.e. a chain of \( c \) curves forming an \( n \)-gon.
- **Type II \( \mathbf{II}_n \)** \( C_p \) consists of \( n+5 \) smooth rational curves meeting with dual graph \( \widetilde{D}_{n+4} \), i.e. a chain of \( n+1 \) curves (multiplicity 2) with two additional curves (multiplicity 1) attached at either end.
- **Type II \( \mathbf{II}_n \)** \( C_p \) is a cuspidal rational curve.
Type III $C_p$ consists of two smooth rational curves meeting with dual graph $\tilde{A}_1$, i.e. they meet at one point to order 2.

Type IV $C_p$ consists of three smooth rational curves meeting with dual graph $\tilde{A}_2$, i.e. they intersect at one point.

Type IV* $C_p$ consists of seven smooth rational curves meeting with dual graph $\tilde{E}_6$

Type III* $C_p$ consists of eight smooth rational curves meeting with dual graph $\tilde{E}_7$

Type II* $C_p$ consists of nine smooth rational curves meeting with dual graph $\tilde{E}_8$

Here, $\tilde{A}_n$ etc. are the extended Dynkin diagrams corresponding to the simple groups $A_n$ etc. We present a rough idea of how to classify these: let the singular fiber be $\sum n_i E_i$.

1. $K \cdot E_i = 0$ for all $i$.
2. If $r = 1$, $E_i^2 = 0$, $p_a(E_i) = 1$.
3. If $r \geq 2$, then for each $1 \leq i \leq r, E_i^2 = -2, E_i \cong \mathbb{P}^1$, and $\sum_{j \neq i} n_j E_j E_i = 2n_i$.

**Proof.** By exercise, $E_i^2 \leq 0$. If $E_i^2 = 0$, then $E_i$ is a multiple of the fiber and the fiber is irreducible. If $E_i^2 < 0$, then $K \cdot E_i < 0$ is not possible (the genus formula shows that $E_i$ would have to be exceptional), so $K \cdot E_i \geq 0$. Then

(5) \[ 0 = 2g(F) - 2 = F \cdot (F + K) = 0 + F \cdot K = \sum n_i (K \cdot E_i) \implies K \cdot E_i = 0 \forall i \]

and $E_i^2 < 0 \implies g(E_i) = 0, E_i^2 = -2, 0 = E_i \cdot F = \sum_{j \neq i} n_j (E_i E_j) - 2n_i$.

by the genus formula, giving the desired result. \qed

We use this information to bound the graphs that can arise and classify the singular fibers (see Silverman, Advanced Topics in the Arithmetic of Elliptic Curves, Ch. IV).