Let $X$ be as from last time, i.e. equipped with maps $f : X \to B, g : X \to \mathbb{P}^1$. Assume $\text{char}(k) \neq 2,3$ and let $S = \{ c \in \mathbb{P}^1 | F_c \text{ is multiple}\}$. If $c \in \mathbb{P}^1 \setminus S, f_c : F'_c \to B$ is an étale morphism. Then we have the map $f^*_c : \text{Pic}^0(B) \to \text{Pic}^0(F'_c)$, and $\text{Pic}^0(F'_c)$ acts canonically on $F'_c$. Thus, we get an action $B \times F'_c \to F'_c$ for each $c \in \mathbb{P}^1 \setminus S$, and thus actions

$$\sigma_0 : B \times g^{-1}(\mathbb{P}^1 \times S) \to g^{-1}(\mathbb{P}^1 \setminus S), \sigma : B \times X \to X$$

Explicitly, if $b \in B, x \in F'_c \subset X$ with $c \in \mathbb{P}^1 \setminus S$, then $b \cdot X = y$, where $f^* \mathcal{O}_B(b - b_0) \otimes \mathcal{O}_{F'_c}(s) = \mathcal{O}_{F'_c}(y)$. Here $b_0$ is a fixed base point on $B$, which acts as the zero element of the elliptic curve $B$. Apply the norm $N_{F'_c/B}$ to get

$$\mathcal{O}_B(nb - nb_0 + f(x)) - \mathcal{O}_B(f(y))$$

where $n = \deg f_c = F_b \cdot F'_c$. We thus obtain commutative diagrams

$$(2) \quad \begin{array}{ccc} X & \xrightarrow{b} & X \\ f \downarrow & & \downarrow f \\ B & \xrightarrow{t_{nb}} & B \end{array}$$

(where $t_{nb}$ is translation by $nb$) and

$$(3) \quad \begin{array}{ccc} B \times X & \xrightarrow{\alpha} & X \\ \text{id}_{B \times X} \downarrow & & \downarrow f \\ B \times B & \xrightarrow{(b,b') \mapsto nb+b'} & B \end{array}$$

Let $B_0 = F_{b_0}$ and $A_n = \text{Ker } n_B : B \to B$ a group subscheme of $B$. We see that the fibers of $f$ are invariant under the action of $A_n$ on $X$. In particular, $A_n$ acts on $B_0$. Denote this by $\alpha : A_n \to \text{Aut } (B_0)$, where $\text{Aut } (B_0)$ is the group scheme of automorphisms of $B_0$. The action of $B$ on $X$ gives $\tau : B \times B_0 \to X$, which
completes the diagram

\[
\begin{array}{ccc}
B \times B_0 & \xrightarrow{\tau} & X \\
& \downarrow f & \\
& B & \\
\end{array}
\]

(4)

Note that we can’t use \( b_0 \) for an arbitrary element of \( B_0 \), since we already used it for a base point of \( B_0 \). So replace it by \( b \in B \) and \( b' \in B_0 \). On can check that \( \tau(b, x) = \tau(b', x') \Leftrightarrow \sigma(b - b', x) = x' \). Thus, \( X \) is isomorphic to the quotient of \( B \times B_0 \) by the action of \( A_n \) given by \( a \cdot (b, b') = (b + a, \alpha(a)(b')) \) for \( a \in A_n, b \in B, b' \in B_0 \). We can substitute the curve \( B/\text{Ker} \)(\( a \)) for \( B \) to get the following theorem:

**Theorem 1.** Every hyperelliptic surface \( X \) has the form \( X = B_1 \times B_0/A \), where \( B_0, B_1 \) are elliptic curves, \( A \) is a finite group subscheme of \( B_1 \), and \( A \) acts on the product \( B_1 \times B_0 \) by \( a(b, b') = (b + a, \alpha(a)(b')) \) for \( a \in A, b \in B_1, b' \in B_0 \), and \( \alpha : A \to \text{Aut} \)(\( B_0 \)) an injective homomorphism. The two elliptic fibrations \( X \) are given by

\[
\begin{align*}
\text{(5)} \quad f : B_1 \times B_0/A \to B_1/A &= B, \\
g : B_1 \times B_0/A \to B_0/\alpha(A) &\cong \mathbb{P}^1
\end{align*}
\]

We can classify these, using the structure of a group of automorphisms of an elliptic curve \( \text{Aut} \)(\( B_0 \)) = \( B_0 \times \text{Aut} \)(\( B_0, 0 \)) (the group of translations and the group of automorphisms fixing \( 0 \) respectively). Explicitly, we have that

\[
\text{(6) } \quad \text{Aut} \)(\( B_0, 0 \)) \cong \begin{cases} 
\mathbb{Z}/2\mathbb{Z} & j(B_0) \neq 0, 1728 \\
\mathbb{Z}/4\mathbb{Z} & j(B_0) = 1728, \quad \text{i.e. } B_0 \cong \{y^2 = x^3 - x\} \\
\mathbb{Z}/6\mathbb{Z} & j(B_0) = 0, \quad \text{i.e. } B_0 \cong \{y^2 = x^3 - 1\}
\end{cases}
\]

Now \( \alpha(A) \) can’t be a subgroup of translations, else \( B_0/\alpha(A) \) would be an elliptic curve, not \( \mathbb{P}^1 \). Let \( a \in A \) be s.t. \( \alpha(a) \) generates the cyclic group \( \alpha(A) \) in \( \text{Aut} \)(\( B_0)/B_0 \cong \text{Aut} \)(\( B_0, 0 \)). It is easy to see that \( \alpha(a) \) must have a fixed point. Choose that point to be the zero point of \( B_0 \). Now \( \alpha(A) \) is abelian, so is a direct product \( A_0 \times \mathbb{Z}/n\mathbb{Z} \). \( A_0 \) is a subgroup of translations of \( B_0 \) and thus a finite subgroup scheme of \( B_0 \). Since \( A_0 \) and \( \alpha(A) \) commute, we must have \( A_0 \subset \{b' \in B_0 | \alpha(a)(b') = b'\} \). We thus have the following possibilities:

(a) \( n = 2 \) \( \implies \) the fixed points are \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \)

(b) \( n = 3 \) \( \implies \) the fixed points are \( \mathbb{Z}/3\mathbb{Z} \)

(c) \( n = 4 \) \( \implies \) the fixed points are \( \mathbb{Z}/2\mathbb{Z} \)

(d) \( n = 6 \) \( \implies \) the fixed points are \( \{0\} \)

We thus obtain the following classification (Bagnera-de Franchis):

(a1) \( (B_1 \times B_0)/(\mathbb{Z}/2\mathbb{Z}), \) with the generator \( a \) of \( \mathbb{Z}/2\mathbb{Z} \subset B_1[2] \) acting on \( B_1 \times B_0 \) by \( a(b_1, b_0) = (b_1 + a, -b_0) \).
(a2) \((B_1 \times B_0)/(\mathbb{Z}/2\mathbb{Z})^2\), with the generators \(a\) and \(g\) of \((\mathbb{Z}/2\mathbb{Z})^2 \subset (B_1[2])^2\) acting by \(a(b_1, b_0) = (b_1 + a, -b_0), g(b_1, b_0) = (b_1 + g, b_0 + c)\) for \(c \in B_0[2]\).

(b1) \((B_1 \times B_0)/(\mathbb{Z}/3\mathbb{Z})\), with the generator \(a\) of \(\mathbb{Z}/3\mathbb{Z} = B_1[3]\) (s.t. \(\alpha(a) = \omega \in \text{Aut}(B_0, 0)\) an automorphism of order 3 [only when \(j(B_0) = 0\)] acting on \(B_1 \times B_0\) by \(a(b_1, b_0) = (b_1 + a, \omega(b_0))\).

(b2) \((B_1 \times B_0)/(\mathbb{Z}/3\mathbb{Z})^2\), with the generators \(a\) and \(g\) of \((\mathbb{Z}/3\mathbb{Z})^2 = (B_1[3])^2\) acting by \(a(b_1, b_0) = (b_1 + a, \omega(b_0)), g(b_1, b_0) = (b_1 + g, b_0 + c)\) for \(c \in B_0[3]\), is fixed by \(\omega\), i.e. \(\omega(c) = c\).

(c1) \((B_1 \times B_0)/(\mathbb{Z}/4\mathbb{Z})\), with the generator \(a\) of \(\mathbb{Z}/4\mathbb{Z} \subset B_1[4]\) (s.t. \(\alpha(a) = i \in \text{Aut}(B_0, 0)\) an automorphism of order 4 [only when \(j(B_0) = 1728\)] acting on \(B_1 \times B_0\) by \(a(b_1, b_0) = (b_1 + a, i(b_0))\).

(c2) \((B_1 \times B_0)/(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})\), with the generators \(a\) and \(g\) of \(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = B_1[4] \times B_1[2]\) acting by \(a(b_1, b_0) = (b_1 + a, i(b_0)), g(b_1, b_0) = (b_1 + g, b_0 + c)\) for \(c \in B_0[2]\).

(d) \((B_1 \times B_0)/(\mathbb{Z}/6\mathbb{Z})\), with the generator \(a\) of \(\mathbb{Z}/6\mathbb{Z} = B_1[6]\) acting on \(B_1 \times B_0\) by \(a(b_1, b_0) = (b_1 + a, -\omega(b_0))\).

## 1. Classification (contd.)

Our first goal is to prove the following theorem:

**Theorem 2.** Let \(X\) be a minimal surface. Then

(a) \(\exists\) an integral curve \(C\) on \(X\) s.t. \(K \cdot C < 0 \iff \kappa(X) = -\infty \iff p_g = p_0 = 0 \iff p_{12} = 0\).

(b) \(K \cdot C = 0\) for all integral curves \(C\) on \(X\) (i.e. \(K \equiv 0\)) \(\iff \kappa(X) = 0 \iff 4K \sim 0\) or \(6K \sim 0 \iff 12K \sim 0\).

(c) \(K^2 = 0, K \cdot C \geq 0\) for all integral curves \(C\) on \(X\), and \(\exists\) an integral curve \(C'\) with \(K \cdot C' > 0 \iff \kappa(X) = 1 \iff K^2 = 0, |4K|\) or \(|6K|\) contains a strictly positive divisor \(\iff K^2 = 0, |12K|\) has a strictly positive divisor.

(d) \(K^2 > 0, K \cdot C \geq 0\) for all integral curves \(C\) on \(X\) \(\iff \kappa(X) = 2\), in which case \(|2K| = \emptyset\).

We already showed that the 4 classes (given by the first clause) are exhaustive and mutually exclusive. We also proved the equivalences in (a). As a preliminary, we need some results on elliptic and quasielliptic fibrations. Recall that an effective divisor \(D = \sum_{i=1}^r n_iE_i > 0\) is said to be of canonical type if \(K_i \cdot E_i = D \cdot E_i = 0\forall i\) (if \(X \to B\) is an elliptic/quasielliptic fibration, then every fiber has this property). If \(D\) is also connected and \(gcd(n_1, \ldots, n_r) = 1\), then we say that \(D\) is an indecomposable curve or a divisor of canonical type.

**Proposition 1.** Let \(D = \sum n_iE_i > 0\) be an indecomposable curve of canonical type on a minimal surface \(X\), and let \(L\) be an invertible \(\mathcal{O}_D\) module. If \(\deg(L \otimes \mathcal{O}_{E_i}) = 0\) for all \(i\), then \(H^0(D, L) \neq 0\) iff \(L \cong \mathcal{O}_D\). Also, \(H^0(D, \mathcal{O}_D) \cong k\).
Proof. It is enough to show that every nonzero section \( s \) of \( H^0(D, L) \) generates \( L \), i.e. gives an isomorphism \( \mathcal{O}_D \cong L \). Then \( H^0(D, \mathcal{O}_D) \) is a field containing \( k \) and is finite dimensional over \( k \). Since \( k \) is algebraically closed by assumption, we have the proposition. So let \( s \in H^0(D, L) \) be nonzero, and let \( s_i = s|_{E_i} \in H^0(E_i, L \otimes \mathcal{O}_{E_i}) \). The fact that \( \deg (L \otimes \mathcal{O}_{E_i}) = 0 \) implies that either \( s_i \) is identically 0 on \( E_i \) or \( s_i \) doesn’t vanish anywhere on \( E_i \) (i.e. it generates \( L \otimes \mathcal{O}_{E_i} \)). If \( s_i \) is identically 0 on \( E_i \), then \( s_j \) must be 0 on \( E_i \) for every \( E_j \) that intersects \( E_i \). This implies that \( s_j \) vanishes at a point of \( E_j \) and thus on all of \( E_j \) for all \( j \) by the connectedness of \( D \). So if \( s \) doesn’t vanish identically on \( E_i \) for all \( i \), then \( s \) doesn’t vanish anywhere on \( D \), and we again have the desired isomorphism.

So suppose that \( s_i \) is identically 0 on \( E_i \) for every \( i \). We’ll show that \( s \neq 0 \) gives a contradiction. Let \( k_i \) be the order of vanishing of \( s_i \) along \( E_i \), \( 1 \leq k \leq n_i \). Whenever \( k_i < n_i \), \( s \) defines a nonzero section of \( L \otimes \mathcal{O}_X(\mathcal{O}_X(-k_iE_i)/\mathcal{O}_X((-k_i+1)E_i)) \). We claim that this section vanishes at every point \( p \in E_i \) to order at least the intersection multiplicity \( (E_i, \sum_{j \neq i} k_j E_j; p) \). To see this, note that locally, if \( E_i \) only intersects one component \( E_j \), \( j \neq i \) at \( p \), we can let \( A = \mathcal{O}_{X,p} \) and \( t_i = 0, t_j = 0 \) cut out \( E_i \) and \( E_j \) respectively at \( p \). We obtain an exact sequence

\[
0 \rightarrow H^0(E_i, L \otimes \mathcal{O}_X(-k_iE_i) \otimes \mathcal{O}_{E_i}) \rightarrow H^0(L \otimes \mathcal{O}_{(k_i+1)E_i}) \rightarrow H^0(L \otimes \mathcal{O}_{K_iE_i})
\]

from the exact sequence

\[
0 \rightarrow \mathcal{O}_X(-k_iE_i) \otimes \mathcal{O}_{E_i} \rightarrow \mathcal{O}_{(k_i+1)E_i} \rightarrow \mathcal{O}_{K_iE_i} \rightarrow 0
\]

after tensoring by \( L \). The local version is

\[
s \in A/(t_i^{m_i}t_j^{n_j})
\]

\[
0 \rightarrow A/t_i \rightarrow A/t_i^{k_i+1} \rightarrow A/t_i^{k_i} \rightarrow 0
\]

We can write \( s = t_i^{k_i} \alpha_i = t_j^{k_j} \alpha_j, \alpha_i, \alpha_j \in A \) since the order of vanishing of \( s \) along \( t_i \) is \( k_i \). Since \( t_i, t_j \) is an \( A \)-regular sequence, we get \( \alpha_i = t_i^{k_j} \beta, \alpha_j = t_i^{k_j} \beta \), for some \( \beta \in A \). The section \( s \) is represented by

\[
t_i^{k_i}t_j^{k_j} \beta = t_j^{k_j} \beta \mod t_i
\]

in \( A/t_i \) to the left of the diagram. Then

\[
\text{ord}_p(t_j^{k_j} \beta) = \dim (A/(t_i, t_j^{k_j} \beta)) \geq \dim (A/(t_i, t_j^{k_j})) = \text{int.mult.}(E_i, k_j E_j; P)
\]
In general, one can use the Chinese remainder theorem to get the inequality for many points $P$. So if $k_i < n_i$ then we have

$$ (t_i, \sum_{j \neq i} k_j E_j) \leq \deg E_i (L \otimes \mathcal{O}_X(-k_i E_i) \otimes \mathcal{O}_{E_i}) $$

$$ \leq \deg (\mathcal{O}_X(-E_i)/\mathcal{O}_X(-2E_i))^{k_i} = -k_i E_i^2 \leq 0 $$

On the other hand, if $k_i = n_i$, then $E_i \cdot D = 0$ gives $E_i \cdot \sum k_j E_j = -(E_i, \sum (n_j - k_j) E_j) \leq 0$ since $k_j \leq n_j$ and $E_i \cdot E_j \geq 0$. So letting $D_1 = \sum k_j E_j$, we have $D_1 \cdot E_i \leq 0$ for all $i$. But

$$(D_1, D) = \sum k_i (E_i, D) = 0$$

$$ \implies D_1 \cdot E_i = 0 \forall i$$

$$ \implies D_1^2 = 0$$

$$ (13) \implies D_1 \text{ is a rational multiple of } D$$

$$ \implies D_1 = D$$

$$ \implies k_i = n_i \forall i \text{ (since } k_i \leq n_i \text{ and } \gcd \{n_i\} = 1)$$

$$ \implies s \equiv 0$$

a contradiction. \qed