4. The Macdonald-Mehta integral

4.1. Finite Coxeter groups and the Macdonald-Mehta integral. Let $W$ be a finite Coxeter group of rank $r$ with real reflection representation $\mathfrak{h}_R$ equipped with a Euclidean $W$-invariant inner product $(\cdot, \cdot)$. Denote by $\mathfrak{h}$ the complexification of $\mathfrak{h}_R$. The reflection hyperplanes subdivide $\mathfrak{h}_R$ into $|W|$ chambers; let us pick one of them to be the dominant chamber and call its interior $D$. For each reflection hyperplane, pick the perpendicular vector $\alpha \in \mathfrak{h}_R$ with $(\alpha, \alpha) = 2$ which has positive inner products with elements of $D$, and call it the positive root corresponding to this hyperplane. The walls of $D$ are then defined by the equations $(\alpha_i, v) = 0$, where $\alpha_i$ are simple roots. Denote by $S$ the set of reflections in $W$, and for a reflection $s \in S$ denote by $\alpha_s$ the corresponding positive root. Let

$$\delta(x) = \prod_{s \in S} (\alpha_s, x)$$

be the corresponding discriminant polynomial. Let $d_i, i = 1, \ldots, r$, be the degrees of the generators of the algebra $\mathbb{C}[\mathfrak{h}]^W$. Note that $|W| = \prod d_i$.

Let $H_{1,c}(W, \mathfrak{h})$ be the rational Cherednik algebra of $W$. Here we choose $c = -k$ as a constant function. Let $M_c = M_c(\mathbb{C})$ be the polynomial representation of $H_{1,c}(W, \mathfrak{h})$, and $\beta_c$ be the contravariant form on $M_c$ defined in Section 3.12. We normalize it by the condition $\beta_c(1, 1) = 1$.

**Theorem 4.1.** (i) *(The Macdonald-Mehta integral)* For $\text{Re}(k) \geq 0$, one has

$$\int_{\mathfrak{h}_R} e^{-(x,x)/2} |\delta(x)|^{2k} dx = \prod_{i=1}^r \frac{\Gamma(1 + kd_i)}{\Gamma(1 + k)}.$$ (4.1)

(ii) Let $b(k) := \beta_c(\delta, \delta)$. Then

$$b(k) = |W| \prod_{i=1}^r \prod_{m=1}^{d_i-1} (kd_i + m).$$

For Weyl groups, this theorem was proved by E. Opdam [Op1]. The non-crystallographic cases were done by Opdam in [Op2] using a direct computation in the rank 2 case (reducing (4.1) to the beta integral by passing to polar coordinates), and a computer calculation by F. Garvan for $H_3$ and $H_4$.

**Example 4.2.** In the case $W = S_n$, we have the following integral (the Mehta integral):

$$(2\pi)^{-n(n-1)/2} \int_{\{x \in \mathbb{R}^n \mid \sum x_i = 0\}} e^{-(x,x)/2} \prod_{i \neq j} |x_i - x_j|^{2k} dx = \prod_{d=2}^n \frac{\Gamma(1 + kd)}{\Gamma(1 + k)}.$$  

In the next subsection, we give a uniform proof of Theorem 4.1 which is given in [E2]. We emphasize that many parts of this proof are borrowed from Opdam’s previous proof of this theorem.

4.2. Proof of Theorem 4.1.

**Proposition 4.3.** The function $b$ is a polynomial of degree at most $|S|$, and the roots of $b$ are negative rational numbers.
Proof. Since $\delta$ has degree $|S|$, it follows from the definition of $b$ that it is a polynomial of degree $\leq |S|$. Suppose that $b(k) = 0$ for some $k \in \mathbb{C}$. Then $\beta_c(\delta, P) = 0$ for any polynomial $P$. Indeed, if there exists $P$ such that $\beta_c(\delta, P) \neq 0$, then there exists such a $P$ which is antisymmetric of degree $|S|$. Then $P$ must be a multiple of $\delta$ which contradicts the equality $\beta_c(\delta, \delta) = 0$.

Thus, $M_c$ is reducible and hence has a singular vector, i.e. a nonzero homogeneous polynomial $f$ of positive degree $d$ living in an irreducible representation $\tau$ of $W$ killed by $y_a$. Applying the element $h = \sum x_a y_a + r/2 + k \sum_{s \in S} s$ to $f$, we get

$$k = -\frac{d}{m_\tau},$$

where $m_\tau$ is the eigenvalue of the operator $T := \sum_{s \in S}(1 - s)$ on $\tau$. But it is clear (by computing the trace of $T$) that $m_\tau \geq 0$ and $m_\tau \in \mathbb{Q}$. This implies that any root of $b$ is negative rational. \hfill $\square$

Denote the Macdonald-Mehta integral by $F(k)$.

**Proposition 4.4.** One has

$$F(k + 1) = b(k)F(k).$$

*Proof.* Let $F = \sum_i y_{a_i}^2/2$. Introduce the Gaussian inner product on $M_c$ as follows:

**Definition 4.5.** The Gaussian inner product $\gamma_c$ on $M_c$ is given by the formula

$$\gamma_c(v, v') = \beta_c(\exp(F)v, \exp(F)v').$$

This makes sense because the operator $F$ is locally nilpotent on $M_c$. Note that $\delta$ is a nonzero $W$-antisymmetric polynomial of the smallest possible degree, so $(\sum y_{a_i}^2)\delta = 0$ and hence

(4.2) $\gamma_c(\delta, \delta) = \beta_c(\delta, \delta) = b(k)$.

For $a \in h$, let $x_a \in h^* \subset H_{1,c}(W, h)$, $y_a \in h \subset H_{1,c}(W, h)$ be the corresponding generators of the rational Cherednik algebra.

**Proposition 4.6.** Up to scaling, $\gamma_c$ is the unique $W$-invariant symmetric bilinear form on $M_c$ satisfying the condition

$$\gamma_c((x_a - y_a)v, v') = \gamma_c(v, y_a v'), \quad a \in h.$$

*Proof.* We have

$$\gamma_c((x_a - y_a)v, v') = \beta_c(\exp(F)(x_a - y_a)v, \exp(F)v') = \beta_c(x_a \exp(F)v, \exp(F)v')$$

$$= \beta_c(\exp(F)v, y_a \exp(F)v') = \beta_c(\exp(F)v, \exp(F)y_a v') = \gamma_c(v, y_a v').$$

Let us now show uniqueness. If $\gamma$ is any $W$-invariant symmetric bilinear form satisfying the condition of the Proposition, then let $\beta(v, v') = \gamma(\exp(-F)v, \exp(-F)v')$. Then $\beta$ is contravariant, so it’s a multiple of $\beta_c$, hence $\gamma$ is a multiple of $\gamma_c$. \hfill $\square$

Now we will need the following known result (see [Du2], Theorem 3.10).
Proposition 4.7. For $\text{Re} \ (k) \geq 0$ we have

\begin{equation}
\gamma_c(f, g) = F(k)^{-1} \int_{b_k} f(x)g(x)d\mu_c(x)
\end{equation}

where

\[ d\mu_c(x) := e^{-(x,x)/2}y(x)^2kdx. \]

Proof. It follows from Proposition 4.6 that $\gamma_c$ is uniquely, up to scaling, determined by the condition that it is $W$-invariant, and $y_a^1 = x_a - y_a$. These properties are easy to check for the right hand side of (4.3), using the fact that the action of $y_a$ is given by Dunkl operators. \qed

Now we can complete the proof of Proposition 4.4. By Proposition 4.7, we have

\[ F(k + 1) = F(k)\gamma_c(\delta, \delta), \]

so by (4.2) we have

\[ F(k + 1) = F(k)b(k). \]

Let

\[ b(k) = b_0 \prod (k + k_i)^{n_i}. \]

We know that $k_i > 0$, and also $b_0 > 0$ (because the inner product $\beta_0$ on real polynomials is positive definite).

Corollary 4.8. We have

\[ F(k) = b_0^k \prod \left( \frac{\Gamma(k + k_i)}{\Gamma(k_i)} \right)^{n_i}. \]

Proof. Denote the right hand side by $F_s(k)$ and let $\phi(k) = F(k)/F_s(k)$. Clearly, $\phi(0) = 1$. Proposition 4.4 implies that $\phi(k)$ is a 1-periodic positive function on $[0, \infty)$. Also by the Cauchy-Schwarz inequality,

\[ F(k)F(k') \geq F\left((k + k')/2\right)^2, \]

so $\log F(k)$ is convex for $k \geq 0$. This implies that $\phi = 1$, since $(\log F_s(k))'' \rightarrow 0$ as $k \rightarrow +\infty$. \qed

Remark 4.9. The proof of this corollary is motivated by the standard proof of the following well known characterization of the $\Gamma$ function.

Proposition 4.10. The $\Gamma$ function is determined by three properties:

(i) $\Gamma(x)$ is positive on $[1, +\infty)$ and $\Gamma(1) = 1$;
(ii) $\Gamma(x + 1) = x\Gamma(x)$;
(iii) $\log \Gamma(x)$ is a convex function on $[1, +\infty)$.

Proof. It is easy to see from the definition $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ that the $\Gamma$ function has properties (i) and (ii); property (iii) follows from this definition and the Cauchy-Schwarz inequality.

Conversely, suppose we have a function $F(x)$ satisfying the above properties, then we have $F(x) = x\Gamma(x)$ for some 1-periodic function $\phi(x)$ with $\phi(x) > 0$. Thus, we have

\[ (\log F)' = (\log \phi)' + (\log \Gamma)'. \]
Since \( \lim_{x \to +\infty} (\log \Gamma)^{\prime\prime} = 0 \), \( (\log F)^{\prime\prime} \geq 0 \), and \( \phi \) is periodic, we have \( (\log \phi)^{\prime\prime} \geq 0 \). Since \( \int_n^{n+1} (\log \phi)^{\prime\prime} dx = 0 \), we see that \( (\log \phi)^{\prime\prime} \equiv 0 \). So we have \( \phi(x) \equiv 1 \).

In particular, we see from Corollary 4.8 and the multiplication formulas for the \( \Gamma \) function that part (ii) of Theorem 4.1 implies part (i).

It remains to establish (ii).

**Proposition 4.11.** The polynomial \( b \) has degree exactly \( |S| \).

**Proof.** By Proposition 4.3, \( b \) is a polynomial of degree at most \( |S| \). To see that the degree is precisely \( |S| \), let us make the change of variable \( s = k^{1/2} y \) in the Macdonald-Mehta integral and use the steepest descent method. We find that the leading term of the asymptotics of \( \log F(k) \) as \( k \to +\infty \) is \( |S| k \log k \). This together with the Stirling formula and Corollary 4.8 implies the statement. \( \square \)

**Proposition 4.12.** The function

\[
G(k) := F(k) \prod_{j=1}^{r} \frac{1 - e^{2\pi i kdj}}{1 - e^{2\pi i k}}
\]

analytically continues to an entire function of \( k \).

**Proof.** Let \( \xi \in D \) be an element. Consider the real hyperplane \( C_t = i t \xi + \mathbb{R}, t > 0 \). Then \( C_t \) does not intersect reflection hyperplanes, so we have a continuous branch of \( \delta(x)^{2k} \) on \( C_t \) which tends to the positive branch in \( D \) as \( t \to 0 \). Then, it is easy to see that for any \( w \in W \), the limit of this branch in the chamber \( w(D) \) will be \( e^{2\pi i \ell(w)} \delta(x)^{2k} \), where \( \ell(w) \) is the length of \( w \). Therefore, by letting \( t = 0 \), we get

\[
(2\pi)^{-r/2} \int_{C_t} e^{-\langle x, x \rangle/2} \delta(x)^{2k} dx = \frac{1}{|W|} F(k) \left( \sum_{w \in W} e^{2\pi i \ell(w)} \right)
\]

(as this integral does not depend on \( t \) by Cauchy’s theorem). But it is well known that

\[
\sum_{w \in W} e^{2\pi i \ell(w)} = \prod_{j=1}^{r} \frac{1 - e^{2\pi i kdj}}{1 - e^{2\pi i k}};
\]

([Hu], p.73), so

\[
(2\pi)^{-r/2} |W| \int_{C_t} e^{-\langle x, x \rangle/2} \delta(x)^{2k} dx = G(k).
\]

Since \( \int_{C_t} e^{-\langle x, x \rangle/2} \delta(x)^{2k} dx \) is clearly an entire function, the statement is proved. \( \square \)

**Corollary 4.13.** For every \( k_0 \in [-1,0] \) the total multiplicity of all the roots of \( b \) of the form \( k_0 - p, p \in \mathbb{Z}_+ \), equals the number of ways to represent \( k_0 \) in the form \( -m/d_i, m = 1, \ldots, d_i - 1 \). In other words, the roots of \( b \) are \( k_{i,m} = -m/d_i - p_{i,m}, 1 \leq m \leq d_i - 1, \) where \( p_{i,m} \in \mathbb{Z}_+ \).

**Proof.** We have

\[
G(k - p) = \frac{F(k)}{b(k - 1) \cdots b(k - p)} \prod_{j=1}^{r} \frac{1 - e^{2\pi i kdj}}{1 - e^{2\pi i k}};
\]
Now plug in \( k = 1 + k_0 \) and a large positive integer \( p \). Since by Proposition 4.12 the left hand side is regular, so must be the right hand side, which implies the claimed upper bound for the total multiplicity, as \( F(1 + k_0) > 0 \). The fact that the bound is actually attained follows from the fact that the polynomial \( b \) has degree exactly \( |S| \) (Proposition 4.11), and the fact that all roots of \( b \) are negative rational (Proposition 4.3).

It remains to show that in fact in Corollary 4.13, \( p_{i,m} = 0 \) for all \( i,m \); this would imply (ii) and hence (i).

**Proposition 4.14.** Identity (4.1) of Theorem 4.1 is satisfied in \( \mathbb{C}[k]/k^2 \).

**Proof.** Indeed, we clearly have \( F(0) = 1 \). Next, a rank 1 computation gives \( F'(0) = -\gamma|S| \), where \( \gamma \) is the Euler constant (i.e. \( \gamma = \lim_{n \to +\infty}(1 + \cdots + 1/n - \log n) \)), while the derivative of the right hand side of (4.1) at zero equals to

\[
-\gamma \sum_{i=1}^{r} (d_i - 1).
\]

But it is well known that

\[
\sum_{i=1}^{r} (d_i - 1) = |S|,
\]

([Hu], p.62), which implies the result. \( \square \)

**Proposition 4.15.** Identity (4.1) of Theorem 4.1 is satisfied in \( \mathbb{C}[k]/k^3 \).

Note that Proposition 4.15 immediately implies (ii), and hence the whole theorem. Indeed, it yields that

\[
(\log F)''(0) = \sum_{i=1}^{r} \sum_{m=1}^{d_i-1} (\log \Gamma)'(m/d_i),
\]

so by Corollary 4.13

\[
\sum_{i=1}^{r} \sum_{m=1}^{d_i-1} (\log \Gamma)'(m/d_i + p_{i,m}) = \sum_{i=1}^{r} \sum_{m=1}^{d_i-1} (\log \Gamma)'(m/d_i),
\]

which implies that \( p_{i,m} = 0 \) since \( (\log \Gamma)' \) is strictly decreasing on \([0, \infty)\).

To prove Proposition 4.15, we will need the following result about finite Coxeter groups.

Let \( \psi(W) = 3|S|^2 - \sum_{i=1}^{r} (d_i^2 - 1) \).

**Lemma 4.16.** One has

\[
(4.4) \quad \psi(W) = \sum_{G \in \mathrm{Par}_2(W)} \psi(G),
\]

where \( \mathrm{Par}_2(W) \) is the set of parabolic subgroups of \( W \) of rank 2.

**Proof.** Let

\[
Q(q) = |W| \prod_{i=1}^{30} \frac{1 - q}{1 - q^{d_i}},
\]
It follows from Chevalley’s theorem that
\[ Q(q) = (1 - q)^r \sum_{w \in \mathcal{W}} \det(1 - qw|_h)^{-1}. \]
Let us subtract the terms for \( w = 1 \) and \( w \in \mathcal{S} \) from both sides of this equation, divide both sides by \((q - 1)^2\), and set \( q = 1 \) (cf. [Hu], p. 62, formula (21)). Let \( W_2 \) be the set of elements of \( W \) that can be written as a product of two different reflections. Then by a straightforward computation we get
\[ \frac{1}{24} \psi(W) = \sum_{w \in W_2} r - \frac{1}{\Tr_h(w)}. \]
In particular, this is true for rank 2 groups. The result follows, as any element \( w \in W_2 \) belongs to a unique parabolic subgroup \( G_w \) of rank 2 (namely, the stabilizer of a generic point \( h^w \), [Hu], p. 22).

**Proof of Proposition 4.15.** Now we are ready to prove the proposition. By Proposition 4.14, it suffices to show the coincidence of the second derivatives of (4.1) at \( k = 0 \). The second derivative of the right hand side of (4.1) at zero is equal to
\[ \frac{\pi^2}{6} \sum_{i=1}^{r} (d_i^2 - 1) + \gamma^2 |S|^2. \]
On the other hand, we have
\[ F''(0) = (2\pi)^{-r/2} \sum_{\alpha, \beta \in \mathcal{S}} \int_{h_S} e^{-\langle x, x \rangle/2} \log \alpha^2(x) \log \beta^2(x) dx. \]
Thus, from a rank 1 computation we see that our job is to establish the equality
\[ (2\pi)^{-r/2} \sum_{\alpha \neq \beta \in \mathcal{S}} \int_{h_S} e^{-\langle x, x \rangle/2} \log \alpha^2(x) \log \beta^2(x) dx = \frac{\pi^2}{6} \left( \sum_{i=1}^{r} (d_i^2 - 1) - 3|S|^2 \right) = -\frac{\pi^2}{6} \psi(W). \]
Since this equality holds in rank 2 (as in this case (4.1) reduces to the beta integral), in general it reduces to equation (4.4) (as for any \( \alpha \neq \beta \in S, s_\alpha \) and \( s_\beta \) are contained in a unique parabolic subgroup of \( W \) of rank 2). The proposition is proved.

4.3. **Application: the supports of \( L_\gamma(C) \).** In this subsection we will use the Macdonald-Mehta integral to computation of the support of the irreducible quotient of the polynomial representation of a rational Cherednik algebra (with equal parameters). We will follow the paper [E3].

First note that the vector space \( \mathfrak{h} \) has a stratification labeled by parabolic subgroups of \( W \). Indeed, for a parabolic subgroup \( W' \subset W \), let \( \mathfrak{h}_{\text{reg}}^{W'} \) be the set of points in \( \mathfrak{h} \) whose stabilizer is \( W' \). Then
\[ \mathfrak{h} = \coprod_{W' \in \text{Par}(W)} \mathfrak{h}_{\text{reg}}^{W'}, \]
where \( \text{Par}(W) \) is the set of parabolic subgroups in \( W \).

For a finitely generated module \( M \) over \( \mathbb{C}[\mathfrak{h}] \), denote the support of \( M \) by \( \text{supp}(M) \).

The following theorem is proved in [Gi1], Section 6 and in [BE] with different method. We will recall the proof from [BE] later.
Theorem 4.17. Consider the stratification of \( \mathfrak{h} \) with respect to stabilizers of points in \( W \). Then the support \( \text{supp}(M) \) of any object \( M \) of \( \mathcal{O}_c(W, \mathfrak{h}) \) in \( \mathfrak{h} \) is a union of strata of this stratification.

This makes one wonder which strata occur in \( \text{supp}(L_c(\tau)) \), for given \( c \) and \( \tau \). In [VW], Varagnolo and Vasserot gave a partial answer for \( \text{supp}(L_c(\tau)) \). Namely, they determined (for \( W \) being a Weyl group) when \( L_c(\mathbb{C}) \) is finite dimensional, which is equivalent to \( \text{supp}(L_c(\mathbb{C})) = 0 \). For the proof (which is quite complicated), they used the geometry affine Springer fibers. Here we will give a different (and simpler) proof. In fact, we will prove a more general result.

Recall that for any Coxeter group \( W \), we have the Poincaré polynomial:

\[
P_W(q) = \sum_{w \in W} q^{\ell(w)} = \prod_{i=1}^{r} \frac{1 - q^{d_i(W)}}{1 - q}, \text{ where } d_i(W) \text{ are the degrees of } W.
\]

Lemma 4.18. If \( W' \subset W \) is a parabolic subgroup of \( W \), then \( P_W \) is divisible by \( P_{W'} \).

Proof. By Chevalley’s theorem, \( \mathbb{C}[^{\mathfrak{h}}] \) is a free module over \( \mathbb{C}[\mathfrak{h}]W \) and \( \mathbb{C}[\mathfrak{h}]W' \) is a direct summand in this module. So \( \mathbb{C}[\mathfrak{h}]W' \) is a projective module, thus free (since it is graded).

Hence, there exists a polynomial \( Q(q) \) such that we have

\[
Q(q)h_{\mathbb{C}[\mathfrak{h}]W}(q) = h_{\mathbb{C}[\mathfrak{h}]W'}(q),
\]

where \( h_V(q) \) denotes the Hilbert series of a graded vector space \( V \). Notice that we have

\[
h_{\mathbb{C}[\mathfrak{h}]W}(q) = \frac{1}{P_W(q)(1-q)^{r}},
\]

so we have

\[
\frac{Q(q)}{P_W(q)} = \frac{1}{P_{W'}(q)}, \text{ i.e. } Q(q) = P_W(q)/P_{W'}(q).
\]

Corollary 4.19. If \( m \geq 2 \) then we have the following inequality:

\[
\#\{i|m \text{ divides } d_i(W)\} \geq \#\{i|m \text{ divides } d_i(W')\}.
\]

Proof. This follows from Lemma 4.18 by looking at the roots of the polynomials \( P_W \) and \( P_{W'} \).

Our main result is the following theorem.

Theorem 4.20. [E3] Let \( c \geq 0 \). Then \( a \in \text{supp}(L_c(\mathbb{C})) \) if and only if

\[
\frac{P_W}{P_{W_a}}(e^{2\pi ic}) \neq 0.
\]

We can obtain the following corollary easily.

Corollary 4.21. (i) \( L_c(\mathbb{C}) \neq M_c(\mathbb{C}) \) if and only if \( c \in \mathbb{Q}_{>0} \) and the denominator \( m \) of \( c \) divides \( d_i \) for some \( i \);

(ii) \( L_c(\mathbb{C}) \) is finite dimensional if and only if \( \frac{P_W}{P_{W'}}(e^{2\pi ic}) = 0 \), i.e., iff

\[
\#\{i|m \text{ divides } d_i(W)\} > \#\{i|m \text{ divides } d_i(W')\},
\]

for any maximal parabolic subgroup \( W' \subset W \).
Remark 4.22. Varagnolo and Vasserot prove that \( L_c(\mathbb{C}) \) is finite dimensional if and only if there exists a regular elliptic element in \( W \) of order \( m \). Case-by-case inspection shows that this condition is equivalent to the combinatorial condition of (2). Also, a uniform proof of this equivalence is given in the appendix to [E3], written by S. Griffeth.

Example 4.23. For type \( A_{n-1} \), i.e., \( W = \mathfrak{S}_n \), we get that \( L_c(\mathbb{C}) \) is finite dimensional if and only if the denominator of \( c \) is \( n \). This agrees with our previous results in type \( A_{n-1} \).

Example 4.24. Suppose \( W \) is the Coxeter group of type \( E_7 \). Then we have the following list of maximal parabolic subgroups and the degrees (note that \( E_7 \) itself is not a maximal parabolic).

<table>
<thead>
<tr>
<th>Subgroups</th>
<th>( E_7 )</th>
<th>( D_6 )</th>
<th>( A_3 \times A_2 \times A_1 )</th>
<th>( A_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degrees</td>
<td>2,6,8,10,12,14,18</td>
<td>2,4,6,6,8,10</td>
<td>2,3,4,2,3,2</td>
<td>2,3,4,5,6,7</td>
</tr>
<tr>
<td>Subgroups</td>
<td>( A_4 \times A_2 )</td>
<td>( E_6 )</td>
<td>( D_5 \times A_1 )</td>
<td>( A_5 \times A_1 )</td>
</tr>
<tr>
<td>Degrees</td>
<td>2,3,4,5,2,3</td>
<td>2,5,6,8,9,12</td>
<td>2,4,5,6,8,2</td>
<td>2,3,4,5,6,2</td>
</tr>
</tbody>
</table>

So \( L_c(\mathbb{C}) \) is finite dimensional if and only if the denominator of \( c \) is 2, 6, 14, 18.

The rest of the subsection is dedicated to the proof of Theorem 4.20. First we recall some basic facts about the Schwartz space and tempered distributions.

Let \( \mathcal{S}(\mathbb{R}^n) \) be the set of Schwartz functions on \( \mathbb{R}^n \), i.e.

\[
\mathcal{S}(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) \mid \forall \alpha, \beta, \sup \{ x^\alpha \partial^\beta f(x) \} < \infty \}.
\]

This space has a natural topology.

A tempered distribution on \( \mathbb{R}^n \) is a continuous linear functional on \( \mathcal{S}(\mathbb{R}^n) \). Let \( \mathcal{S}'(\mathbb{R}^n) \) denote the space of tempered distributions.

We will use the following well known lemma.

Lemma 4.25.  
(i) \( \mathbb{C}[x] e^{-x^2/2} \subset \mathcal{S}(\mathbb{R}^n) \) is a dense subspace.

(ii) Any tempered distribution \( \xi \) has finite order, i.e., \( \exists N = N(\xi) \) such that if \( f \in \mathcal{S}(\mathbb{R}^n) \) satisfying \( f = df = \cdots = d^{N-1}f = 0 \) on \( \text{supp} \xi \), then \( \langle \xi, f \rangle = 0 \).

Proof of Theorem 4.20. Recall that on \( M_c(\mathbb{C}) \), we have the Gaussian form \( \gamma_c \) from Section 4.2. We have for \( \text{Re} \,(c) \leq 0 \),

\[
\gamma_c(P, Q) = \frac{(2\pi)^{-r/2}}{F_W(-c)} \int_{\mathbb{R}^r} e^{-x^2/2} |\delta(x)|^{-2c} P(x)Q(x)dx,
\]

where \( P, Q \) are polynomials and

\[
F_W(k) = (2\pi)^{-r/2} \int_{\mathbb{R}^r} e^{-x^2/2} |\delta(x)|^{2k}dx
\]

is the Macdonald-Mehta integral.

Consider the distribution:

\[
\zeta^W_c = \frac{(2\pi)^{-r/2}}{F_W(-c)} |\delta(x)|^{-2c}.
\]

It is well-known that this distribution is meromorphic in \( c \) (Bernstein’s theorem). Moreover, since \( \gamma_c(P, Q) \) is a polynomial in \( c \) for any \( P \) and \( Q \), this distribution is in fact holomorphic in \( c \in \mathbb{C} \).
\[
\text{supp } (\xi_c^W) = \{ a \in \mathfrak{h} \cap \frac{F_{W_a}}{F_W}(-c) \neq 0 \} = \{ a \in \mathfrak{h} \cap \frac{P_W}{P_{W_a}}(e^{2\pi i c}) \neq 0 \}
\]
\[
= \{ a \in \mathfrak{h} \cap \#\{i | \text{denominator of } c \text{ divides } d_i(W) \} \}
= \#\{i | \text{denominator of } c \text{ divides } d_i(W_a) \}.
\]

Proof. First note that the last equality follows from the product formula for the Poincaré polynomial, and the second equality from the Macdonald-Mehta identity. Now let us prove the first equality.

Look at \(\xi_c^W\) near \(a \in \mathfrak{h}\). Equivalently, we can consider
\[
\xi_c^W(x + a) = \frac{(2\pi)^{-r/2}}{F_W(-c)} |\delta(x + a)|^{-2c}
\]
with \(x\) near 0. We have
\[
\delta_W(x + a) = \prod_{s \in S} \alpha_s(x + a) = \prod_{s \in S} (\alpha_s(x) + \alpha_s(a))
\]
\[
= \prod_{s \in S \cap W_a} \alpha_s(x) \cdot \prod_{s \in S \setminus S \cap W_a} (\alpha_s(x) + \alpha_s(a))
\]
\[
= \delta_{W_a}(x) \cdot \Psi(x),
\]
where \(\Psi\) is a nonvanishing function near \(a\) (since \(\alpha_s(a) \neq 0\) if \(s \notin S \cap W_a\)).

So near \(a\), we have
\[
\xi_c^W(x + a) = \frac{F_{W_a}}{F_W}(-c) \cdot \xi_c^W(x) \cdot |\Psi|^{-2c},
\]
and the last factor is well defined since \(\Psi\) is nonvanishing. Thus \(\xi_c^W(x)\) is nonzero near \(a\) if and only if \(\frac{F_{W_a}}{F_W}(-c) \neq 0\) which finishes the proof.

\[\square\]

Proposition 4.27. For \(c \geq 0\),
\[
\text{supp } (\xi_c^W) = \text{supp } L_c(\mathbb{C})_{\mathbb{R}},
\]
where the right hand side stands for the real points of the support.

Proof. Let \(a \notin \text{supp } L_c(\mathbb{C})\) and assume \(a \in \text{supp } \xi_c^W\). Then we can find a \(P \in J_c(\mathbb{C}) = \ker \gamma_c\) such that \(P(a) \neq 0\). Pick a compactly supported test function \(\phi \in C_c^\infty(\mathfrak{h}_{\mathbb{R}})\) such that \(P\) does not vanish anywhere on \(\text{supp } \phi\), and \((\xi_c^W, \phi) \neq 0\) (this can be done since \(P(a) \neq 0\) and \(\xi_c^W\) is nonzero near \(a\)). Then we have \(\phi/P \in \mathcal{S}(\mathfrak{h}_{\mathbb{R}})\). Thus from Lemma 4.25 (i) it follows that there exists a sequence of polynomials \(P_n\) such that
\[
P_n(x)e^{-x^2/2} \to \frac{\phi}{P} \text{ in } \mathcal{S}(\mathfrak{h}_{\mathbb{R}}), \text{ when } n \to \infty.
\]
So \(PP_n e^{-x^2/2} \to \phi\) in \(\mathcal{S}(\mathfrak{h}_{\mathbb{R}})\), when \(n \to \infty\).

But we have \((\xi_c^W, PP_n e^{-x^2/2}) = \gamma_c(P, P_n) = 0\) which is a contradiction. This implies that \(\text{supp } \xi_c^W \subset (\text{supp } L_c(\mathbb{C}))_{\mathbb{R}}\).

To show the opposite inclusion, let \(P\) be a polynomial on \(\mathfrak{h}\) which vanishes identically on \(\text{supp } \xi_c^W\). By Lemma 4.25 (ii), there exists \(N\) such that \((\xi_c^W, P^N(x)Q(x)e^{-x^2/2}) = 0\). Thus,
for any polynomial $Q$, $\gamma_c(P^N, Q) = 0$, i.e. $P^N \in \text{Ker} \gamma_c$. Thus, $P|_{\text{supp} L_c(C)} = 0$. This implies the required inclusion, since $\text{supp} \xi^W_c$ is a union of strata.

Theorem 4.20 follows from Proposition 4.26 and Proposition 4.27.

4.4. **Notes.** Our exposition in Sections 4.1 and 4.2 follows the paper [E2]; Section 4.3 follows the paper [E3].