SOLUTIONS TO EXERCISES

CHAPTER I

A. Manifolds

A.2. If \( p_1, p_2 \in M \) are sufficiently close within a coordinate neighborhood \( U \), there exists a diffeomorphism mapping \( p_1 \) to \( p_2 \) and leaving \( M - U \) pointwise fixed. Now consider a curve segment \( \gamma(t) (0 \leq t \leq 1) \) in \( M \) joining \( p \) to \( q \). Let \( t^* \) be the supremum of those \( t \) for which there exists a diffeomorphism of \( M \) mapping \( p \) on \( \gamma(t) \). The initial remark shows first that \( t^* > 0 \), next that \( t^* = 1 \), and finally that \( t^* \) is reached as a maximum.

A.3. The “only if” is obvious and “if” follows from the uniqueness in Prop. 1.1. Now let \( \mathcal{C} = C^\infty(\mathbb{R}) \) where \( \mathbb{R} \) is given the ordinary differentiable structure. If \( n \) is an odd integer, let \( \mathcal{E}^n \) denote the set of functions \( x \rightarrow f(x^n) \) on \( \mathbb{R} \), \( f \in \mathcal{C} \) being arbitrary. Then \( \mathcal{E}^n \) satisfies \( \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \). Since \( \mathcal{E}^n \neq \mathcal{E}^m \) for \( n \neq m \), the corresponding \( \mathcal{E}^n \) are all different.

A.4. (i) If \( d\Phi \cdot X = Y \) and \( f \in C^\infty(N) \), then \( X(f \circ \Phi) = (Yf) \circ \Phi \in \mathcal{C}_0 \). On the other hand, suppose \( X\mathcal{E}_0 \subset \mathcal{C}_0 \). If \( F \in \mathcal{C}_0 \), then \( F = g \circ \Phi \) where \( g \in C^\infty(N) \) is unique. If \( f \in C^\infty(N) \), then \( X(f \circ \Phi) = g \circ \Phi \) (\( g \in C^\infty(N) \) unique), and \( f \rightarrow g \) is a derivation, giving \( Y \).

(ii) If \( d\Phi \cdot X = Y \), then \( Y_{\sigma(p)} = d\Phi_{\sigma(p)}(X_p) \), so necessity follows. Suppose \( d\Phi_{\gamma}(M_p) = N_{\sigma(p)} \) for each \( p \in M \). Define for \( r \in N \), \( Y_r = d\Phi_{\gamma}(X_p) \) if \( r = \Phi(p) \). In order to show that \( Y : r \rightarrow Y_r \) is differentiable we use (by virtue of Theorem 15.5) coordinates around \( p \) and around \( r = \Phi(p) \) such that \( \Phi \) has the expression \( (x_1, ..., x_m) \rightarrow (x_1, ..., x_n) \). Writing

\[
X = \sum_{i=1}^{m} a_i(x_1, ..., x_m) \frac{\partial}{\partial x_i},
\]

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we have for \( q \) sufficiently near \( p \)

\[
d\Phi_q(X_q) = \sum_{i=1}^{n} a_i(x_1(q), \ldots, x_n(q)) \left( \frac{\partial}{\partial x_i} \right)_{\phi(q)},
\]

so condition (1) implies that for \( 1 \leq i \leq n \), \( a_i \) is constant in the last \( m - n \) arguments. Hence

\[
Y = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n, x_{n+1}(p), \ldots, x_m(p)) \frac{\partial}{\partial x_i}.
\]

(iii) \( f \in C^\infty(N) \) if and only if \( f \circ \psi \in C^\infty(R) \). If \( f(x) = x^2 \), then \( f \circ \psi(x) = x \), \( (f' \circ \psi)(x) = 2x \), so \( f \in C^\infty(N) \), \( f' \notin C^\infty(N) \). Hence \( f \circ \Phi \in \mathcal{B}_0 \), but \( X(f \circ \Phi) \notin \mathcal{B}_0 \); so by (i), \( X \) is not projectable.

**A.5.** Obvious.

**A.6.** Use Props. 15.2 and 15.3 to shrink the given covering to a new one; then use the result of Exercise A.1 to imitate the proof of Theorem 1.3.

**A.7.** We can assume \( M = R^m \), \( p = 0 \), and that \( X_0 = (\partial/\partial t_1)_0 \) in terms of the standard coordinate system \( \{t_1, \ldots, t_m\} \) on \( R^m \). Consider the integral curve \( \varphi(0, c_2, \ldots, c_m) \) of \( X \) through \( (0, c_2, \ldots, c_m) \). Then the mapping \( \psi : (c_1, \ldots, c_m) \rightarrow \varphi(c_1(0, c_2, \ldots, c_m)) \) is \( C^\infty \) for small \( c_i \), \( \psi(0, c_2, \ldots, c_m) = (0, c_2, \ldots, c_m) \), so

\[
d\psi_0 \left( \frac{\partial}{\partial c_i} \right) = \left( \frac{\partial}{\partial t_i} \right)_{0} \quad (i > 1).
\]

Also

\[
d\psi_0 \left( \frac{\partial}{\partial c_1} \right) = \left( \frac{\partial \varphi_{c_1}}{\partial c_1} \right)(0) = X_0 = \left( \frac{\partial}{\partial t_1} \right)_{0}.
\]

Thus \( \psi \) can be inverted near 0, so \( \{c_1, \ldots, c_m\} \) is a local coordinate system. Finally, if \( \epsilon = (c_1, \ldots, c_m) \),

\[
\left( \frac{\partial}{\partial c_1} \right)_{\psi(\epsilon)} f = \left( \frac{\partial (f \circ \psi)}{\partial c_1} \right)_{\epsilon}
\]

\[
= \lim_{h \to 0} \frac{1}{h} [f(\varphi_{c_1+h}(0, c_2, \ldots, c_m)) - f(\varphi_{c_1}(0, c_2, \ldots, c_m))]
\]

\[
= (Xf)(\psi(\epsilon))
\]

so \( X = \partial/\partial c_1 \).
A.8. Let \( f \in C^\omega(M) \). Writing \( \sim \) below when in an equality we omit terms of higher order in \( s \) or \( t \), we have

\[
f(\psi_-s(\psi_t(\psi_t(o)))) - f(o)
= f(\psi_-s(\psi_t(\psi_t(o)))) - f(\psi_-s(\psi_t(\psi_t(o))))
+ f(\psi_t(\psi_t(o))) - f(\psi_t(\psi_t(o)))
+ f(\psi_t(\psi_t(o))) - f(\psi_t(\psi_t(o)))
\sim -t(Yf)(\psi_t(\psi_t(o))) + \frac{1}{2}s(2f)(\psi_t(\psi_t(o)))
- s(Xf)(\psi_t(\psi_t(o))) + \frac{1}{2}s(2f)(\psi_t(\psi_t(o)))
+ s(Xf)(\psi_t(\psi_t(o))) - \frac{1}{2}s(2f)(\psi_t(\psi_t(o)))
\sim st(XYf)(\psi_t(\psi_t(o)))\). 

This last expression is obtained by pairing off the 1st and 5th term, the 3rd and 7th, the 2nd and 6th, and the 4th and 8th. Hence

\[
f(y(t^2)) - f(o) = t^2([X, Y])f(o) + O(t^3).
\]

A similar proof is given in Faber [1].

B. The Lie Derivative and the Interior Product

B.1. If the desired extension of \( \theta(X) \) exists and if \( C : \mathcal{D}_1(M) \to C^\omega(M) \) is the contraction, then (i), (ii), (iii) imply

\[
(\theta(X)\omega)(Y) = X(\omega(Y)) - \omega([X, Y]),
\]

for \( X, Y \in \mathcal{D}_1(M) \). Thus we define \( \theta(X) \) on \( \mathcal{D}_1(M) \) by this relation and note that \( \theta(X)\omega)(fY) = f(\theta(X)(\omega)(Y)) \) \( (f \in C^\omega(M)) \), so \( \theta(X) \mathcal{D}_1(M) \subseteq \mathcal{D}_1(M) \). If \( U \) is a coordinate neighborhood with coordinates \( \{x_1, \ldots, x_m\} \), \( \theta(X) \) induces an endomorphism of \( C^\omega(U) \), \( \mathcal{D}_1(U) \), and \( \mathcal{D}_1(U) \). Putting \( X_i = \partial/\partial x_i \), \( \omega_j = dx_j \), each \( T \in \mathcal{D}_1(U) \) can be written

\[
T = \sum T_{(i), (j)}X_i \otimes \cdots \otimes X_i \otimes \omega_j \otimes \cdots \otimes \omega_j,
\]

with unique coefficients \( T_{(i), (j)} \in C^\omega(U) \). Now \( \theta(X) \) is uniquely extended to \( \mathcal{D}(U) \) satisfying (i) and (ii). Property (iii) is then verified by induction on \( r \) and \( s \). Finally, \( \theta(X) \) is defined on \( \mathcal{D}(M) \) by the condition \( \theta(X)T \mid U = \theta(X)(T \mid U) \) (vertical bar denoting restriction) because as in the proof of Theorem 2.5 this condition is forced by the requirement that \( \theta(X) \) should be a derivation.

B.2. The first part being obvious, we just verify \( \phi \cdot \omega = (\phi^{-1})^* \omega \). We may assume \( \omega \in \mathcal{D}_1(M) \). If \( X \in \mathcal{D}_1(M) \) and \( C \) is the contraction \( X \otimes \omega \to \omega(X) \), then \( C \phi \circ \phi \) implies \( (\phi \cdot \omega)(X) = \phi(\omega(X)) \) \( = ((\phi^{-1})^* \omega)(X) \).
B.3. The formula is obvious if \( T = f \in C^\infty(M) \). Next let \( T = Y \in \mathcal{D}_1(M) \). If \( f \in C^\infty(M) \) and \( q \in M \), we put \( F(t, q) = f(g_t \cdot q) \) and have

\[
F(t, q) - F(0, q) = t \int_0^1 \left( \frac{\partial F}{\partial t} \right)(st, q) \, ds = t \, h(t, q),
\]

where \( h \in C^\infty(R \times M) \) and \( h(0, q) = (Xf)(q) \). Then

\[
(g_t \cdot Y)_\rho f = (Y(f \circ g_t))(g_t^{-1} \cdot \rho) = (Yf)(g_t^{-1} \cdot \rho) + t(Yh)(t, g_t^{-1} \cdot \rho)
\]

so

\[
\lim_{t \to 0} \frac{1}{t} (Y - g_t \cdot Y)_\rho f = (XYf)(\rho) - (YXf)(\rho),
\]

so the formula holds for \( T \in \mathcal{D}_1(M) \). But the endomorphism \( T \to \lim_{t \to 0} t^{-1}(T - g_t \cdot T) \) has properties (i), (ii), and (iii) of Exercise B.1; it coincides with \( \theta(X) \) on \( C^\infty(M) \) and on \( \mathcal{D}_1(M) \), hence on all of \( \mathcal{D}(M) \) by the uniqueness in Exercise B.1.

B.4. For (i) we note that both sides are derivations of \( \mathcal{D}(M) \) commuting with contractions, preserving type, and having the same effect on \( \mathcal{D}_1(M) \) and on \( C^\infty(M) \). The argument of Exercise B.1 shows that they coincide on \( \mathcal{D}(M) \).

(ii) If \( \omega \in \mathcal{D}(M) \), \( Y_1, ..., Y_r \in \mathcal{D}_1(M) \), then by B.1,

\[
(\theta(X)\omega)(Y_1, ..., Y_r) = X(\omega(Y_1, ..., Y_r)) - \sum_t \omega(Y_1, ..., [X, Y_t], ..., Y_r)
\]

so \( \theta(X) \) commutes with \( A \).

(iii) Since \( \theta(X) \) is a derivation of \( \mathfrak{g}(M) \) and \( d \) is a skew-derivation (that is, satisfies (iv) in Theorem 2.5), the commutator \( \theta(X)d - d\theta(X) \) is also a skew-derivation. Since it vanishes on \( f \) and \( df \) (\( f \in C^\infty(M) \)), it vanishes identically (cf. Exercise B.1). For B.1–B.4, cf. Palais [3].

B.5. This is done by the same method as in Exercise B.1.

B.6. For (i) we note that by (iii) in Exercise B.5, \( i(X)^2 \) is a derivation. Since it vanishes on \( C^\infty(M) \) and \( \mathcal{D}_1(M) \), it vanishes identically; (ii) follows by induction; (iii) follows since both sides are skew-derivations which coincide on \( C^\infty(M) \) and on \( \mathfrak{g}(M) \); (iv) follows because both sides are derivations which coincide on \( C^\infty(M) \) and on \( \mathfrak{g}(M) \).

C. Affine Connections

C.1. \( M \) has a locally finite covering \( \{ U_a \}_{a \in A} \) by coordinate neighborhoods \( U_a \). On \( U_a \) we construct an arbitrary Riemannian structure \( g_a \). If \( 1 = \sum_a \varphi_a \) is a partition of unity subordinate to the covering, then \( \sum_a \varphi_a g_a \) gives the desired Riemannian structure on \( M \).
C.2. If \( \Phi \) is an affine transformation and we write \( d\Phi(\partial/\partial x_i) = \sum_i a_{ij} \partial/\partial x_j \), then conditions \( \nabla_1 \) and \( \nabla_2 \) imply that each \( a_{ij} \) is a constant. If \( A \) is the linear transformation \( (a_{ij}) \), then \( \Phi \circ A^{-1} \) has differential \( I \), hence is a translation \( B \), so \( \Phi(X) = AX + B \). The converse is obvious.

C.3. We have \( \Phi^* \omega^i_j = \sum_k (\Gamma^i_{kj} \circ \Phi) \Phi^* \omega^k_j \), so by (5'), (6), (7) in \( \S 8 \)

\[ \Phi^* \omega^i_j = \sum_k (\Gamma^i_{kj} \circ \Phi)(a_k \, dt + t \, da_k) = 0. \]

This implies that \( \Gamma^i_{kj} \equiv 0 \) in normal coordinates, which is equivalent to the result stated in the exercise.

C.4. A direct verification shows that the mapping \( \delta: \theta \rightarrow \sum_i \omega_i \wedge \nabla x_i(\theta) \) is a skew-derivation of \( \mathfrak{a}(M) \) and that it coincides with \( d \) on \( C^\infty(M) \). Next let \( \theta \in \mathfrak{a}_1(M) \), \( X, Y \in \mathfrak{d}(M) \). Then, using (5), \( \S 7 \),

\[ 2 \delta \theta(X, Y) = 2 \sum_i (\omega_i \wedge \nabla x_i(\theta))(X, Y) \]
\[ = \sum_i \omega_i(X) \nabla x_i(\theta)(Y) - \omega_i(Y) \nabla x_i(\theta)(X) \]
\[ = \nabla x(\theta)(Y) - \nabla y(\theta)(X) \]
\[ = X \cdot \theta(Y) - \theta(\nabla x(Y)) - Y \cdot \theta(X) + \theta(\nabla y(X)), \]

which since the torsion is 0 equals

\[ X \theta(Y) - Y \cdot \theta(X) - \theta([X, Y]) = 2 \, d\theta(X, Y). \]

Thus \( \delta = d \) on \( \mathfrak{a}_1(M) \), hence by the above on all of \( \mathfrak{a}(M) \).

C.5. No; an example is given by a circular cone with the vertex rounded off.

C.6. Using Props. 11.3 and 11.4 we obtain a mapping \( \varphi: M \rightarrow N \) such that \( d\varphi_p \) is an isometry for each \( p \in M \). Thus \( \varphi(M) \subset N \) is an open subset. Each geodesic in the manifold \( \varphi(M) \) is indefinitely extendable, so \( \varphi(M) \) is complete, whence \( \varphi \) maps \( M \) onto \( N \). Now Lemma 13.4 implies that \( (M, \varphi) \) is a covering space of \( N \), so \( M \) and \( N \) are isometric.

D. Submanifolds

D.1. Let \( I: G_\varphi \rightarrow M \times N \) denote the identity mapping and \( \pi: M \times N \rightarrow M \) the projection onto the first factor. Let \( m \in M \) and \( Z \in \mathfrak{g}_\varphi(m, \varphi(m)) \) such that \( dI_m(Z) = 0 \). Then \( Z = (d\varphi)_m(X) \) where \( X \in M_m \). Thus \( d\pi \circ dI \circ d\varphi(X) = 0 \). But since \( \pi \circ I \circ \varphi \) is the identity mapping, this implies \( X = 0 \), so \( Z = 0 \) and \( I \) is regular.

D.3. Consider the figure 8 given by the formula

$$y(t) = (\sin 2t, \sin t) \quad (0 \leq t \leq 2\pi).$$

Let $f(s)$ be an increasing function on $\mathbb{R}$ such that

$$\lim_{s \to -\infty} f(s) = 0, \quad f(0) = \pi, \quad \lim_{s \to \infty} f(s) = 2\pi.$$

Then the map $s \to y(f(s))$ is a bijection of $\mathbb{R}$ onto the figure 8. Carrying the manifold structure of $\mathbb{R}$ over, we get a submanifold of $\mathbb{R}^2$ which is closed, yet does not carry the induced topology. Replacing $y$ by $\delta$ given by $\delta(t) = (-\sin 2t, \sin t)$, we get another manifold structure on the figure.

D.4. Suppose $\text{dim } M < \text{dim } N$. Using the notation of Prop. 3.2, let $W$ be a compact neighborhood of $p$ in $M$ and $W \subset U$. By the countability assumption, countably many such $W$ cover $M$. Thus by Lemma 3.1, Chapter II, for $N$, some such $W$ contains an open set in $N$; contradiction.

D.5. For each $m \in M$ there exists by Prop. 3.2 an open neighborhood $V_m$ of $m$ in $N$ and an extension of $g$ from $V_m \cap M$ to a $C^\infty$ function $G_m$ on $V_m$. The covering $\{V_m\}_{m \in M}$, $N - M$ of $N$ has a countable locally finite refinement $V_1, V_2, \ldots$. Let $\varphi_1, \varphi_2, \ldots$ be the corresponding partition of unity. Let $\varphi_{i_1}, \varphi_{i_2}, \ldots$ be the subsequence of the $(\varphi_i)$ whose supports intersect $M$, and for each $\varphi_{i_p}$ choose $m_p \in M$ such that $\text{supp}(\varphi_{i_p}) \subset V_{m_p}$. Then $\sum_p G_m \varphi_{i_p}$ is the desired function $G$.

D.6. The “if” part is contained in Theorem 14.5 and the “only if” part is immediate from (2), Chapter V, §6.

E. Curvature

E.1. If $(r, \theta)$ are polar coordinates of a vector $X$ in the tangent space $M_p$, the inverse of the map $(r, \theta) \to \text{Exp}_p X$ gives the “geodesic polar coordinates” around $p$. Since the geodesics from $p$ intersect sufficiently small circles around $p$ orthogonally (Lemma 9.7), the Riemannian structure has the form $g = dr^2 + \varphi(r, \theta)^2 d\theta^2$. In these coordinates the Riemannian measure $f \to \int f \sqrt{g} \, dx_1 \ldots \, dx_n$ and the Laplace-Beltrami operator are, respectively, given by

$$f \to \int \int f(r, \theta) \varphi(r, \theta) \, dr \, d\theta,$$

and

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \varphi^{-1} \frac{\partial \varphi}{\partial r} \frac{\partial f}{\partial r} + \varphi^{-1} \frac{\partial^2 f}{\partial \theta} \left( \varphi^{-1} \frac{\partial f}{\partial \theta} \right).$$
In particular
\[ \Delta(\log r) = -\frac{1}{r^2} + \frac{1}{r\varphi} \frac{\partial \varphi}{\partial r} . \]

On the other hand, if \((x, y)\) are the normal coordinates of \(\text{Exp}_p X\) such that
\[ r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}, \]
then, since \(r\, dr = x\, dx + y\, dy, r^2\, d\theta = x\, dy - y\, dx,\)
\[ g = r^{-4}[(x^2r^2 + y^2\varphi^2)\, dx^2 + 2xy(r^2 - \varphi^2)\, dx\, dy + (y^2r^2 + x^2\varphi^2)\, dy^2] \]
so since the coefficients are smooth near \((x, y) = (0, 0)\) \(\varphi^2\) has the form \(^{+}\)
\[ \varphi^2 = r^2 + cr^4 + \ldots, \]
where \(c = c(p)\) is a constant. But then
\[ \lim_{r \to 0} \Delta(\log r) = c(p). \]

On the other hand,
\[ A(r) = \int_0^r \int_0^{2\pi} \varphi(t, \theta)\, dt\, d\theta, \]
so using the definition in §12 we find \(K = -3c(p)\) as stated.

This result is stated in Klein [1], p. 219, without proof (with opposite sign).

\textbf{E.2.} Let \(X = \partial/\partial x_1\) and \(Y = \partial/\partial x_2\) so \(\gamma_e\) is formed by integral curves of \(X, Y, -X, -Y.\)

\[ \text{Let } p = p_0 = (0, 0, \ldots, 0) \]
\[ p_1 = (\epsilon, 0, \ldots, 0) \]
\[ p_2 = (\epsilon, \epsilon, \ldots, 0) \]
\[ p_3 = (0, \epsilon, \ldots, 0) \]

and \(\tau_{ij}\) the parallel transport from \(p_j\) to \(p_i\) along \(\gamma_e.\) Let \(T\) be any vector field on \(M,\) and write \(T_i = T_{p_i}.\) Then
\[ \tau_{03}^2 \tau_{21} T_{10} T_0 - T_0 \]
\[ = (\tau_{03}^2 \tau_{21} T_{10} T_0 - \tau_{03} \tau_{21} T_1) + (\tau_{03} \tau_{21} T_1 - \tau_{03} \tau_{32} T_2) \]
\[ + (\tau_{32} T_2 - \tau_{03} T_3) + (\tau_{03} T_3 - T_0). \]

We use Theorem 7.1 and write \( \sim \) when we omit terms of higher order in \( \epsilon \). Then our expression is

\[
\sim \tau_{03} \tau_{23} \left[ -\epsilon (\nabla_x T)_1 + \frac{1}{2} \epsilon^2 (\nabla_x^2 T)_1 \right] \\
+ \tau_{03} \tau_{23} \left[ -\epsilon (\nabla_y T)_2 + \frac{1}{2} \epsilon^2 (\nabla_y^2 T)_2 \right] \\
- \tau_{03} \tau_{23} \left[ -\epsilon (\nabla_x T)_2 + \frac{1}{2} \epsilon^2 (\nabla_x^2 T)_2 \right] \\
- \tau_{03} \left[ -\epsilon (\nabla_y T)_3 + \frac{1}{2} \epsilon^2 (\nabla_y^2 T)_3 \right].
\]

Combining now the 1st and 5th term, 2nd and 6th term, etc., this expression reduces to

\[
\sim e^2 \tau_{03} \tau_{23} (\nabla_y (\nabla_x (T)))_3 - e^2 \tau_{03} (\nabla_x (\nabla_y (T)))_3
\]

which, since \([X, Y] = 0\), reduces to

\[
\sim e^2 \tau_{03} (R(Y, X)T)_3 \sim e^2 (R(Y, X)T)_0.
\]

This proof is a simplification of that of Faber [1]. See Laugwitz [1], §10 for another version of the result. For curvature and holonomy groups, see e.g. Ambrose and Singer [2].

**F. Surfaces**

**F.1.** Let \( Z \) be a vector field on \( S \) and \( \bar{X}, \bar{Y}, \bar{Z} \) vector fields on a neighborhood of \( s \) in \( \mathbb{R}^3 \) extending \( X, Y, \) and \( Z, \) respectively. The inner product \( \langle \cdot, \cdot \rangle \) on \( \mathbb{R}^3 \) induces a Riemannian structure \( g \) on \( S \). If \( \nabla \) and \( \nabla' \) denote the corresponding affine connections on \( \mathbb{R}^3 \) and \( S, \) respectively, we deduce from (2), §9

\[
\langle \bar{Z}, \nabla_{\bar{X}} (\bar{Y})_s \rangle = g(Z, \nabla_x (Y)_s).
\]

But

\[
\nabla_{\bar{X}} (\bar{Y})_s = \lim_{t \to 0} \frac{1}{t} (Y_{s(t)} - Y_s),
\]

so we obtain \( \nabla = \nabla' \); in particular \( \nabla' \) is an affine connection on \( S \).

**F.2.** Let \( \delta (u, v) \rightarrow (u, v) \) be local coordinates on \( S \) and if \( g \) denotes the Riemannian structure on \( S, \) put

\[
E = g \frac{\partial}{\partial u}, \quad F = g \frac{\partial}{\partial u}, \quad G = g \frac{\partial}{\partial v}.
\]
Let \( r(u, v) \) denote the vector from 0 to the point \( s(u, v) \). Subscripts denoting partial derivatives, \( r_u \) and \( r_v \), span the tangent space at \( s(u, v) \), and we may take the orientation such that

\[
\xi_{s(u,v)} = \frac{r_u \times r_v}{|r_u \times r_v|},
\]

\( \times \) denoting the cross product. We have

\[
\dot{\gamma}_S = r_u \dot{u} + r_v \dot{v}
\]

and

\[
\ddot{\gamma}_S = r_{uu} \dot{u}^2 + 2r_{uv} \dot{u} \dot{v} + r_{vv} \dot{v}^2 + r_u \ddot{u} + r_v \ddot{v},
\]

whence

\[
\begin{align*}
r_{uu} \cdot r_u &= \frac{1}{2} E_u, & r_{uv} \cdot r_u &= \frac{1}{2} E_v, & r_{vv} \cdot r_u &= \frac{1}{2} G_u, \\
r_{uv} \cdot r_v &= \frac{3}{2} G_u, & r_{uu} \cdot r_v &= F_u - \frac{1}{2} E_u, & r_{uv} \cdot r_v &= F_v - \frac{1}{2} E_v.
\end{align*}
\]

From this it is clear that the geodesic curvature can be expressed in terms of \( \dot{u}, \dot{v}, \ddot{u}, \ddot{v}, E, F, G, \) and their derivatives, and therefore has the invariance property stated.

**F.3.** We first recall that under the orthogonal projection \( P \) of \( \mathbb{R}^3 \) on the tangent space \( S_{\gamma_S(t)} \) the curve \( P \circ \gamma_S \) has curvature in \( \gamma_S(t) \) equal to the geodesic curvature of \( \gamma_S \) at \( \gamma_S(t) \). So in order to avoid discussing developable surfaces we define the rolling in the problem as follows. Let \( \pi = S_{s_S(t_0)} \) and let \( t \to \gamma_n(t) \) be the curve in \( \pi \) such that

\[
\gamma_n(t_0) = \gamma_S(t_0), \quad \dot{\gamma}_n(t_0) = \dot{\gamma}_S(t_0)
\]

\((t - t_0)\) is the arc-parameter measured from \( \gamma_n(t_0) \)) and such that the curvature of \( \gamma_n \) at \( \gamma_n(t) \) is the geodesic curvature of \( \gamma_S \) at \( \gamma_S(t) \). The rolling is understood as the family of isometries \( S_{\gamma_S(t)} \to \pi_{\gamma_n(t)} \) of the tangent planes such that the vector \( \dot{\gamma}_S(t) \) is mapped onto \( \dot{\gamma}_n(t) \). Under these maps a Euclidean parallel family of unit vectors along \( \gamma_n \) corresponds to a family \( Y(t) \in S_{\gamma_S(t)} \). We must show that this family is parallel in the sense of (1), §5. Let \( \tau \) denote the angle between \( \dot{\gamma}_S(t) \) and \( Y(t) \). Then

\[
\ddot{\tau}(t) = -\text{curvature of } \gamma_n \text{ at } \gamma_n(t) = -\text{geodesic curvature of } \gamma_S \text{ at } \gamma_S(t) = -(\xi \times \dot{\gamma}_S \cdot \ddot{\gamma}_S(t)).
\]

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We can choose the coordinates \((u, v)\) near \(\gamma_s(t_0)\) such that for \(t\) close to \(t_0\)

\[
u(\gamma_s(t)) = t, \quad v(\gamma_s(t)) = \text{const.}, \quad g_{\gamma_s}(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) = 0.
\]

(For example, let \(r \to \delta(t)\) be a geodesic in \(S\) starting at \(\gamma_s(t)\) perpendicular to \(\gamma_S\); small pieces of these geodesics fill up (disjointly) a neighborhood of \(\gamma_s(t_0)\); the mapping \(\delta(t) \to (t, r)\) is a coordinate system with the desired properties.) Writing \(Y(t) = Y^1(t)r_u + Y^2(t)r_v\) (using notation from previous exercise), we have

\[
Y^1(t) = \cos \tau(t), \quad Y^2(t) = G^{-1/2} \sin \tau(t) \tag{1}
\]

and shall now verify (2), §5. By (2), §9 we have

\[
2 \sum_i g_{ik} \Gamma^i_{ij} = \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_k} g_{ij}.
\]

On the curve \(\gamma_S\) we have \(E \equiv 1, F \equiv 0\), so

\[
\Gamma^1_{11} = 0, \quad \Gamma^1_{12} = -\frac{E_v}{2G}, \quad \Gamma^1_{12} = \frac{E_v}{2},
\]

\[
\Gamma^1_{22} = F_v - \frac{G_u}{2}, \quad \Gamma^1_{22} = \frac{G_u}{2G}, \quad \Gamma^1_{12} = \frac{G_u}{2G}.
\]

Thus we must verify

\[
\dot{Y}^1 + \frac{1}{2} E_v Y^2 = 0, \quad \dot{Y}^2 - \frac{E_v}{2G} Y^1 + \frac{G_u}{2G} Y^2 = 0. \tag{2}
\]
But using formulas from Exercise F.2 we find

\[
\dot{t}(t) = -(\xi \times \gamma_s \cdot \gamma_s)(t) = \frac{1}{2}(G^{-1/2}E_v)(\gamma_s(t))
\]

and now equations (2) follow directly from (1).

G. The Hyperbolic Plane

1. (i) and (ii) are obvious. (iii) is clear since

\[
\frac{x'(t)^2}{(1 - x(t)^2)^2} \leq \frac{x'(t)^2 + y'(t)^2}{(1 - x(t)^2 - y(t)^2)^2}
\]

where \( \gamma(t) = (x(t), y(t)) \). For (iv) let \( z \in D, u \in D_z \), and let \( z(t) \) be a curve with \( z(0) = z, z'(0) = u \). Then

\[
d\varphi_z(u) = \left\{ \frac{d}{dt} \varphi(z(t)) \right\}_{t=0} = \frac{z'(0)}{bz + d}
\]

and \( g(d\varphi(u), d\varphi(u)) = g(u, u) \) now follows by direct computation. Now (v) follows since \( \varphi \) is conformal and maps lines into circles. The first relation in (vi) is immediate; and writing the expression for \( d(0, x) \) as a cross ratio of the points \(-1, 0, x, 1\), the expression for \( d(z_1, z_2) \) follows since \( \varphi \) in (iv) preserves cross ratio. For (vii) let \( \tau \) be any isometry of \( D \). Then there exists a \( \varphi \) as in (iv) such that \( \varphi \tau^{-1} \) leaves the \( x \)-axis pointwise fixed. But then \( \varphi \tau^{-1} \) is either the identity or the complex conjugation \( z \rightarrow \bar{z} \). For (viii) we note that if \( r = d(0, z) \), then \(|z| = \tanh r\); so the formula for \( g \) follows from (ii). Part (ix) follows from

\[
v = \frac{1 - |z|^2}{|z - i|^2}, \quad dw = -2 \frac{dz}{(z - i)^2}, \quad d\bar{w} = -2 \frac{d\bar{z}}{(|\bar{z} + i|^2}.
\]