CHAPTER II

A. On the Geometry of Lie Groups

A.1. (i) follows from \( \exp \text{Ad}(x)tX = x \exp tX x^{-1} = L(x) R(x^{-1}) \exp tX \) for \( X \in \mathfrak{g}, \ t \in \mathbb{R} \). For (ii) we note \( J(x \exp tX) = \exp(-tX) x^{-1} \), so \( dJ_x(dL(x)eX) = -dR(x^{-1})eX \). For (iii) we observe for \( X_0, \ Y_0 \in \mathfrak{g} \)

\[ \Phi(g \exp tX_0, h \exp sY_0) = g \exp tX_0 h \exp sY_0 = gh \exp t \text{Ad}(h^{-1}) X_0 \exp sY_0, \]

(Continued on next page.)
so
\[ d\Phi(dL(g)X_0, dL(h)Y_0) = dL(gh)(\text{Ad}(h^{-1})X_0 + Y_0). \]

Putting \( X = dL(g)X_0, \ Y = dL(h)Y_0, \) the result follows from (i).

**A.2.** Suppose \( \gamma(t_1) = \gamma(t_2) \) so \( \gamma(t_2 - t_1) = e. \) Let \( L > 0 \) be the smallest number such that \( \gamma(L) = e. \) Then \( \gamma(t + L) = \gamma(t) \gamma(L) = \gamma(t). \)

If \( \tau_L \) denotes the translation \( t \to t + L, \) we have \( \gamma \circ \tau_L = \gamma, \) so
\[ \dot{\gamma}(0) = d\gamma \left( \frac{d}{dt} \right)_0 = d\gamma \left( \frac{d}{dt} \right)_L = \dot{\gamma}(L). \]

**A.3.** The curve \( \sigma \) satisfies \( \sigma(t + L) = \sigma(t), \) so in A.2, \( \dot{\sigma}(0) = \dot{\sigma}(L). \)

**A.4.** Let \( (p_n) \) be a Cauchy sequence in \( G/H. \) Then if \( d \) denotes the distance, \( d(p_n, p_m) \to 0 \) if \( m, n \to \infty. \) Let \( B_{e}(o) \) be a relatively compact ball of radius \( \varepsilon > 0 \) around the origin \( o = \{H\} \) in \( G/H. \) Select \( N \) such that \( d(p_n, p_m) < \frac{1}{4}\varepsilon \) for \( m \geq N \) and select \( g \in G \) such that \( g \cdot p_N = o. \) Then \( (g \cdot p_m) \) is a Cauchy sequence inside the compact ball \( B_{e}(o), \) hence it, together with the original sequence, is convergent.

**A.5.** For \( X \in \mathfrak{g} \) let \( \tilde{X} \) denote the corresponding left invariant vector field on \( G. \) From Prop. 1.4 we know that (i) is equivalent to \( \nabla_{\tilde{X}}(Z) = 0 \) for all \( Z \in \mathfrak{g}. \) But by (2), §9 in Chapter I this condition reduces to
\[ \langle Z, [\tilde{X}, Z] \rangle = 0 \quad (X, Z \in \mathfrak{g}) \]
which is clearly equivalent to (ii). Next (iii) follows from (ii) by replacing \( X \) by \( X + Z. \) But (iii) is equivalent to \( \text{Ad}(G) \)-invariance of \( B \) so \( Q \) is right invariant. Finally, the map \( J : x \to x^{-1} \) satisfies \( J = R(g^{-1}) \circ J \circ L(g^{-1}), \) so \( dJ_g = dR(g^{-1})_e \circ dJ_e \circ dL(g^{-1})_g. \) Since \( dJ_e \) is automatically an isometry, (v) follows.

**A.6.** Assuming first the existence of \( \nabla, \) consider the affine transformation \( \sigma : g \to \exp \frac{1}{2}Yg^{-1} \exp \frac{1}{2}Y \) of \( G \) which fixes the point \( \exp \frac{1}{2}Y \) and maps \( \gamma_1, \) the first half of \( \gamma, \) onto the second half, \( \gamma_2. \) Since
\[ \sigma = L(\exp \frac{1}{2}Y) \circ J \circ L(\exp -\frac{1}{2}Y), \]
we have \( d\sigma_{\exp \frac{1}{2}Y} = -I. \) Let \( X^*(t) \in G_{\exp \frac{1}{2}Y} (0 \leq t \leq 1) \) be the family of vectors parallel with respect to \( \gamma \) such that \( X^*(0) = X. \) Then \( \sigma \) maps \( X^*(s) \) along \( \gamma_1 \) into a parallel field along \( \gamma_2 \) which must be the field \(-X^*(t)\) because \( d\sigma(X^*(\frac{1}{2})) = -X^*(\frac{1}{2}). \) Thus the map \( \sigma \circ J = L(\exp \frac{1}{2}Y) \circ R(\exp \frac{1}{2}Y) \) sends \( X \) into \( X^*(1), \) as stated in part (i). Part (ii) now follows from Theorem 7.1, Chapter I, and part (iii) from Prop. 1.4. Now (iv) follows from (ii) and the definition of \( T \) and \( R. \)
Finally, we prove the existence of \( \nabla \). As remarked before Prop. 1.4, the equation \( \nabla_X(Y) = \frac{1}{2}[\bar{X}, \bar{Y}] \ (X, Y \in g) \) defines uniquely a left invariant affine connection \( \nabla \) on \( G \). Since \( \bar{X}^{R(g)} = (\text{Ad}(g^{-1})X)^{\sim} \), we get

\[
\nabla_{\bar{X}}^{R(g)}(\bar{Y}^{R(g)}) = \frac{1}{2}\{\text{Ad}(g^{-1})[X, Y]\}^{\sim} = (\nabla_X(\bar{Y}))^{R(g)};
\]

this we generalize to any vector fields \( Z, Z' \) by writing them in terms of \( \bar{X}_i \) (1 \( \leq i \leq n \)). Next

\[
\nabla_JX(J\bar{Y}) = J(\nabla_X(\bar{Y})). \tag{1}
\]

Since both sides are right invariant vector fields, it suffices to verify the equation at \( e \). Now \( J\bar{X} = -\bar{X} \) where \( \bar{X} \) is right invariant, so the problem is to prove

\[
(\nabla_X(\bar{Y}))_e = -\frac{1}{2}[X, Y].
\]

For a basis \( X_1, \ldots, X_n \) of \( g \) we write \( \text{Ad}(g^{-1})Y = \sum_i f_i(g)X_i \). Since \( \bar{Y}_g = dR(g)Y = dL(g)\text{Ad}(g^{-1})Y \), it follows that \( \bar{Y} = \sum_i f_i\bar{X}_i \), so using \( \nabla_0 \) and Lemma 4.2 from Chapter I, \( \S 4 \),

\[
(\nabla_X(\bar{Y}))_e = (\nabla_X(\bar{Y}))_e = \sum_i (Xf_i)_e X_i + \frac{1}{2} \sum_i f_i(e)[\bar{X}, \bar{X}]_e
\]

Since \( (Xf_i)(e) = \{(d/dt) f_i(\exp tX)\}_t=0 \) and since

\[
\left\{ \frac{d}{dt} \text{Ad}(\exp(-tX))(Y) \right\}_{t=0} = -[X, Y],
\]

the expression on the right reduces to \( -[X, Y] + \frac{1}{2}[X, Y] \), so (1) follows. As before, (1) generalizes to any vector fields \( Z, Z' \).

The connection \( \nabla \) is the \( 0 \)-connection of Cartan-Schouten [1].

**B. The Exponential Mapping**

**B.1.** At the end of \( \S 1 \) it was shown that \( GL(2, \mathbb{R}) \) has Lie algebra \( \mathfrak{gl}(2, \mathbb{R}) \), the Lie algebra of all \( 2 \times 2 \) real matrices. Since \( \det(e^{tx}) = \)
Prop. 2.7 shows that \( \mathfrak{sl}(2, \mathbb{R}) \) consists of all 2 \( \times \) 2 real matrices of trace 0. Writing

\[
X = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}
\]

a direct computation gives for the Killing form

\[
B(X, X) = 8(a^2 + bc) = 4 \text{Tr}(XX),
\]
whence \( B(X, Y) = 4 \text{Tr}(XY) \), and semisimplicity follows quickly.

Part (i) is obtained by direct computation. For (ii) we consider the equation

\[
e^X = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad (\lambda \in \mathbb{R}, \quad \lambda \neq 1).
\]

**Case 1**: \( \lambda > 0 \). Then \( \det X < 0 \). In fact \( \det X = 0 \) implies

\[
I + X = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},
\]

so \( b = c = 0 \), so \( a = 0 \), contradicting \( \lambda \neq 1 \). If \( \det X > 0 \), we deduce quickly from (i) that \( b = c = 0 \), so \( \det X = -a^2 \), which is a contradiction. Thus \( \det X < 0 \) and using (i) again we find the only solution

\[
X = \begin{pmatrix} \log \lambda & 0 \\ 0 & -\log \lambda \end{pmatrix}.
\]

**Case 2**: \( \lambda = -1 \). For \( \det X > 0 \) put \( \mu = (\det X)^{1/2} \). Then using (i) the equation amounts to

\[
\cos \mu + (\mu^{-1} \sin \mu)a = -1, \quad (\mu^{-1} \sin \mu)b = 0,
\]

\[
\cos \mu - (\mu^{-1} \sin \mu)a = -1, \quad (\mu^{-1} \sin \mu)c = 0.
\]

These equations are satisfied for

\[
\mu = (2n + 1)\pi \quad (n \in \mathbb{Z}), \quad \det X = -a^2 - bc = (2n + 1)^2 \pi^2.
\]

This gives infinitely many choices for \( X \) as claimed.

**Case 3**: \( \lambda < 0, \lambda \neq -1 \). If \( \det X = 0 \), then (i) shows \( b = c = 0 \), so \( a = 0 \); impossible. If \( \det X > 0 \) and we put \( \mu = (\det X)^{1/2} \), (i) implies

\[
\cos \mu + (\mu^{-1} \sin \mu)a = \lambda, \quad (\mu^{-1} \sin \mu)b = 0,
\]

\[
\cos \mu - (\mu^{-1} \sin \mu)a = \lambda^{-1}, \quad (\mu^{-1} \sin \mu)c = 0.
\]
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Since $\lambda \neq \lambda^{-1}$, we have $\sin \mu \neq 0$. Thus $b = c = 0$, so $\det X = -a^2$, which is impossible. If $\det X < 0$ and we put $\mu = (-\det X)^{1/2}$, we get from (i) the equations above with $\sin$ and $\cos$ replaced by $\sinh$ and $\cosh$. Again $b = c = 0$, so $\det X = -a^2 = -\mu^2$; thus $a = \pm \mu$, so

$$\cosh \mu \pm \sinh \mu = \lambda, \quad \cosh \mu \mp \sinh \mu = \lambda^{-1},$$

contradicting $\lambda < 0$. Thus there is no solution in this case, as stated.

B.2. The Killing form on $\mathfrak{sl}(2, \mathbb{R})$ provides a bi-invariant pseudo-Riemannian structure with the properties of Exercise A.5. Thus (i) follows from Exercise B.1. Each $g \in SL(2, \mathbb{R})$ can be written $g = k \rho$ where $k \in SO(2)$ and $\rho$ is positive definite. Clearly $k = \exp T$ where $T \in \mathfrak{sl}(2, \mathbb{R})$; and using diagonalization, $\rho = \exp X$ where $X \in \mathfrak{sl}(2, \mathbb{R})$. The formula $g = \exp T \exp X$ proves (ii).

B.3. Follow the hint.

B.4. Considering one-parameter subgroups it is clear that $g$ consists of the matrices

$$X(a, b, c) = \begin{pmatrix} 0 & c & 0 & a \\ -c & 0 & 0 & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (a, b, c \in \mathbb{R}).$$

Then $[X(a, b, c), X(a_1, b_1, c_1)] = X(cb_1 - c_1 b, c_1 a - ca_1, 0)$, so $g$ is readily seen to be solvable. A direct computation gives

$$\exp X(a, b, c) = \begin{pmatrix} \cos c & \sin c & 0 & c^{-1}(a \sin c - b \cos c + b) \\ -\sin c & \cos c & 0 & c^{-1}(b \sin c + a \cos c - a) \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Thus $\exp X(a, b, 2\pi)$ is the same point in $G$ for all $a, b \in \mathbb{R}$, so $\exp$ is not injective. Similarly, the points in $G$ with $\gamma = n2\pi \ (n \in \mathbb{Z})$ $a^2 + b^2 > 0$ are not in the range of $\exp$. This example occurs in Auslander and MacKenzie [1]; the exponential mapping for a solvable group is systematically investigated in Dixmier [2].

B.5. Let $N_0$ be a bounded star-shaped open neighborhood of $0 \in g$ which $\exp$ maps diffeomorphically onto an open neighborhood $N_e$ of $e$ in $G$. Let $N^* = \exp(\frac{1}{2}N_0)$. Suppose $S$ is a subgroup of $G$ contained in $N^*$, and let $s \neq e$ in $S$. Then $s = \exp X \ (X \in \frac{1}{2}N_0)$. Let $k \in \mathbb{Z}^+$ be such that $X, 2X, \ldots, kX \in \frac{1}{2}N_0$ but $(k + 1)X \notin \frac{1}{2}N_0$. Since $N_0$ is star-shaped, $(k + 1)X \in N_0^*$; but since $s^{k+1} \in N^*$, we have $s^{k+1} = \exp Y, Y \in \frac{1}{2}N_0$. Since $\exp$ is one-to-one on $N_0$, $(k + 1)X = Y \in \frac{1}{2}N_0$, which is a contradiction.
C. Subgroups and Transformation Groups

C.1. The proofs given in Chapter X for \( SU^*(2n) \) and \( Sp(n, \mathbb{C}) \) generalize easily to the other subgroups.

C.2. Let \( G \) be a commutative connected Lie group, \((\bar{G}, \pi)\) its universal covering group. By facts stated during the proof of Theorem 1.11, \( \bar{G} \) is topologically isomorphic to a Euclidean group \( R^p \). Thus \( G \) is topologically isomorphic to a factor group of \( R^p \) and by a well-known theorem on topological groups (e.g. Bourbaki [1], Chap. VII) this factor group is topologically isomorphic to \( R^n \times T^m \). Thus by Theorem 2.6, \( G \) is analytically isomorphic to \( R^n \times T^m \).

For the last statement let \( \bar{\gamma} \) be the closure of \( \gamma \) in \( H \). By the first statement and Theorem 2.3, \( \bar{\gamma} = R^n \times T^m \) for some \( n, m \in \mathbb{Z}^+ \). But \( \gamma \) is dense in \( \bar{\gamma} \), so either \( n = 1 \) and \( m = 0 \) (\( \gamma \) closed) or \( n = 0 \) (\( \gamma \) compact).

C.3. By Theorem 2.6, \( I \) is analytic and by Lemma 1.12, \( dI \) is injective. Q.E.D.

C.4. The mapping \( \psi_g \) turns \( g \cdot N_0 \) into a manifold which we denote by \( (g \cdot N_0)_x \). Similarly, \( \psi_{g'} \) turns \( g' \cdot N_0 \) into a manifold \( (g' \cdot N_0)_y \). Thus we have two manifolds \( (g \cdot N_0 \cap g' \cdot N_0)_x \) and \( (g' \cdot N_0 \cap g' \cdot N_0)_y \) and must show that the identity map from one to the other is analytic. Consider the analytic section maps

\[ \sigma_g : (g \cdot N_0)_x \to G, \quad \sigma_{g'} : (g' \cdot N_0)_y \to G \]

defined by

\[ \sigma_g(g \exp(x_1X_1 + \ldots + x_rX_r) \cdot p_0) = g \exp(\pi_1X_1 + \ldots + \pi_rX_r), \]
\[ \sigma_{g'}(g' \exp(y_1X_1 + \ldots + y_rX_r) \cdot p_0) = g' \exp(y_1X_1 + \ldots + y_rX_r), \]

and the analytic map

\[ J_g : \pi^{-1}(g \cdot N_0) \to (g \cdot N_0)_x \times H \]

given by

\[ J_g(z) = (\pi(z), [\sigma_g(\pi(z))]^{-1}z). \]

Furthermore, let \( P : (g \cdot N_0)_x \times H \to (g \cdot N_0)_x \) denote the projection on the first component. Then the identity mapping

\[ I : (g \cdot N_0 \cap g' \cdot N_0)_y \to (g \cdot N_0 \cap g' \cdot N_0)_x \]

can be factored:

\[ (g \cdot N_0 \cap g' \cdot N_0)_y \xrightarrow{\sigma_{g'}} \pi^{-1}(g \cdot N_0) \xrightarrow{J_g} (g \cdot N_0)_x \times H \xrightarrow{P} (g \cdot N_0)_x. \]

\[ \dagger \text{See "Some Details," p. 586.} \]
In fact, if \( p \in g \cdot N_0 \cap g' \cdot N_0 \), we have
\[
p = g \exp(x_1X_1 + \ldots + x_rX_r) \cdot p_0 = g' \exp(y_1X_1 + \ldots + y_rX_r) \cdot p_0,
\]
so for some \( h \in H \),
\[
P(J_\sigma(\alpha(g)(p))) = P(J_\sigma(g' \exp(y_1X_1 + \ldots + y_rX_r))) = P(g' \exp(y_1X_1 + \ldots + y_rX_r), h) = P(g \exp(x_1X_1 + \ldots + x_rX_r), h) = g \exp(x_1X_1 + \ldots + x_rX_r) \cdot p_0.
\]
Thus \( I \) is composed of analytic maps so is analytic, as desired.

**C.5.** The subgroup \( H = G_2 \) of \( G \) leaves \( p \) fixed is closed, so \( G/H \) is a manifold. The map \( I : G/H \rightarrow M \) given by \( I(gH) = g \cdot p \) gives a bijection of \( G/H \) onto the orbit \( G \cdot p \). Carrying the differentiable structure over on \( G \cdot p \) by means of \( I \), it remains to prove that \( I : G/H \rightarrow M \) is everywhere regular. Consider the maps on the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\pi} & G/H \\
\downarrow & & \downarrow I \\
G/H & \xrightarrow{\beta} & M
\end{array}
\]

where \( \pi(g) = gH \), \( \beta(g) = g \cdot p \) so \( \beta = I \circ \pi \). If we restrict \( \pi \) to a local cross section, we can write \( I = \beta \circ \pi^{-1} \) on a neighborhood of the origin in \( G/H \). Thus \( I \) is \( C^\infty \) near the origin, hence everywhere. Moreover, the map \( d\beta_e : g \rightarrow M_0 \) has kernel \( \mathfrak{h} \), the Lie algebra of \( H \) (cf. proof of Prop. 4.3). Since \( d\pi_e \) maps \( g \) onto \( (G/H)_H \) with kernel \( \mathfrak{h} \) and since \( d\beta_e = dI_H \circ d\pi_e \), we see that \( dI_H \) is one-to-one. Finally, if \( T(g) \) denotes the diffeomorphism \( m \rightarrow g \cdot m \) of \( M \), we have \( I = T(g) \circ I \circ \tau(g^{-1}) \), whence
\[
dI_{gH} = dT(g)_p \circ dI_H \circ d\tau(g^{-1})_{pH},
\]
so \( I \) is everywhere regular.

**C.6.** By local connectedness each component of \( G \) is open. It acquires an analytic structure from that of \( G_0 \) by left translation. In order to show the map \( \varphi : (x, y) \rightarrow xy^{-1} \) analytic at a point \( (x_0, y_0) \in G \times G \) let \( G_1 \) and \( G_2 \) denote the components of \( G \) containing \( x_0 \) and \( y_0 \), respectively. If \( \varphi_0 = \varphi \mid G_0 \times G_0 \) and \( \psi = \varphi \mid G_1 \times G_2 \), then
\[
\psi = L(x_0y_0^{-1}) \circ I(y_0) \circ \varphi_0 \circ L(x_0^{-1}, y_0^{-1}),
\]
where \( I(y_0)(x) = y_0xy_0^{-1} \) (\( x \in G_0 \)). Now \( I(y_0) \) is a continuous automorphism of the Lie group \( G_0 \), hence by Theorem 2.6, analytic; so the expression for \( \psi \) shows that it is analytic.

**C.8.** If \( N \) with the indicated properties exists we may, by translation, assume it passes through the origin \( o = \{H\} \) in \( M \). Let \( L \) be the subgroup \( \{g \in G : g \cdot N = N\} \). If \( g \in G \) maps \( o \) into \( N \), then \( gN \cap N \neq \emptyset \); so by assumption, \( gN = N \). Thus \( L = \pi^{-1}(N) \) where \( \pi : G \to G/\mathbf{H} \) is the natural map. Using Theorem 15.5, Chapter I we see that \( L \) can be given the structure of a submanifold of \( G \) with a countable basis and by the transitivity of \( G \) on \( M \), \( L \cdot o = N \). By C.7, \( L \) has the desired property. For the converse, define \( N = L \cdot o \) and use Prop. 4.4 or Exercise C.5. Clearly, if \( gN \cap N \neq \emptyset \), then \( g \in L \), so \( gN = N \).

For more information on the primitivity notion which goes back to Lie see e.g. Golubitsky [1].

**D. Closed Subgroups**

**D.1.** \( \mathbb{R}^2/\Gamma \) is a torus (Exercise C.2), so it suffices to take a line through \( 0 \) in \( \mathbb{R}^2 \) whose image in the torus is dense.

**D.2.** \( g \) has an \( \text{Int}(g) \)-invariant positive definite quadratic form \( Q \). The proof of Prop. 6.6 now shows \( g = g + g' \) (\( 3 = \text{center of } g, g' = [g, g] \) compact and semisimple). The groups \( \text{Int}(g) \) and \( \text{Int}(g') \) are analytic subgroups of \( GL(g) \) with the same Lie algebra so coincide.

**D.3.** We have

\[
\alpha_{\theta, 1}(c_1, c_2, s) = (c_1, e^{2\pi i/3}c_2, s);
\]

\[
(a_1, a_2, r)(c_1, c_2, s)(a_1, a_2, r)^{-1} = (a_1(1 - e^{2\pi i s}) + c_1e^{2\pi i r}, a_2(1 - e^{2\pi i s}) + c_2e^{2\pi i r}, s),
\]

so \( \alpha_{\theta, 1} \) is not an inner automorphism, and \( A_{\theta, 1} \notin \text{Int}(g) \). Now let \( s_n \to 0 \) and let \( t_n = hs_n + hn \). Select a sequence \( (n_k) \subset \mathbb{Z} \) such that \( hn_k \to 1 \) (mod 1) (Kronecker's theorem), and let \( \tau_k \) be the unique point in \([0, 1)\) such that \( t_{n_k} - \tau_k \in \mathbb{Z} \). Putting \( s_k = s_{n_k}, t_k = t_{n_k} \), we have

\[
\alpha_{s_k, t_k} = \alpha_{s_k, t_k} \to \alpha_{0, 1}.
\]

**Note:** \( G \) is a subgroup of \( H \times H \) where \( H = (\frac{1}{c}, 0), c \in \mathbb{C}, |c| = 1 \).

**E. Invariant Differential Forms**

**E.1.** The affine connection on \( G \) given by \( \nabla_\mathbf{X}(\mathbf{Y}) = \frac{1}{2}[\mathbf{X}, \mathbf{Y}] \) is torsion free; and by (5), §7, Chapter I, if \( \omega \) is a left invariant 1-form,

\[
\nabla_\mathbf{X}(\omega)(\mathbf{Y}) = -\omega(\nabla_\mathbf{X}(\mathbf{Y})) = -\frac{1}{2}\omega(\theta(\mathbf{X})(\mathbf{Y})) = \frac{1}{2}(\theta(\mathbf{X})\omega)(\mathbf{Y}),
\]
so $\nabla_{X}(\omega) = \frac{1}{2} \theta(X)(\omega)$ for all left invariant forms $\omega$. Now use Exercise C.4 in Chapter I.

**E.2.** The first relation is proved as (4), §7. For the other we have $g^{t}g = I$, so $(dg)^{t}g + g^{t}(dg) = 0$. Hence $(g^{-1}dg) + (dg)(g^{-1}) = 0$ and $\Omega + 4\Omega = 0$.

For $U(n)$ we find similarly for $\Omega = g^{-1}dg$,

$$d\Omega + \Omega \wedge \Omega = 0, \quad \Omega + 4\Omega = 0.$$  

For $Sp(n) \subset U(2n)$ we recall that $g \in Sp(n)$ if and only if

$$g^{t}g = I_{2n}, \quad gJ_{n}g = J_{n}$$  

(cf. Chapter X). Then the form $\Omega = g^{-1}dg$ satisfies

$$d\Omega + \Omega \wedge \Omega = 0, \quad \Omega + 4\Omega = 0, \quad \Omega J_{n} + J_{n}4\Omega = 0.$$  

**E.3.** A direct computation gives

$$g^{-1}dg = \begin{pmatrix} 0 & dx & dz - x \, dy \\ 0 & 0 & dy \\ 0 & 0 & 0 \end{pmatrix}$$

and the result follows.