A. Manifolds

2. Let $M$ be a connected manifold and $p$, $q$ two points in $M$. Then there exists a diffeomorphism $\Phi$ of $M$ onto itself such that $\Phi(p) = q$.

3. Let $M$ be a Hausdorff space and let $\delta$ and $\delta'$ be two differentiable structures on $M$. Let $\mathcal{G}$ and $\mathcal{G}'$ denote the corresponding sets of $C^\infty$ functions. Then $\delta = \delta'$ if and only if $\mathcal{G} = \mathcal{G}'$.

Deduce that the real line $\mathbb{R}$ with its ordinary topology has infinitely many different differentiable structures.

4. Let $\Phi$ be a differentiable mapping of a manifold $M$ onto a manifold $N$. A vector field $X$ on $M$ is called projectable (Koszul [1]) if there exists a vector field $Y$ on $N$ such that $d\Phi \cdot X = Y$.

(i) Show that $X$ is projectable if and only if $X\mathcal{G}_0 \subset \mathcal{G}_0$ where $\mathcal{G}_0 = \{f \circ \Phi : f \in C^\infty(N)\}$.

(ii) A necessary condition for $X$ to be projectable is that

\[ d\Phi_p(X_p) = d\Phi_q(X_q) \tag{1} \]

whenever $\Phi(p) = \Phi(q)$. If, in addition, $d\Phi_p(M_p) = N_{\Phi(p)}$ for each $p \in M$, this condition is also sufficient.

(iii) Let $M = \mathbb{R}$ with the usual differentiable structure and let $N$ be the topological space $\mathbb{R}$ with the differentiable structure obtained by requiring the homeomorphism $\psi : x \rightarrow x^{1/3}$ of $M$ onto $N$ to be a diffeomorphism. In this case the identity mapping $\Phi : x \rightarrow x$ is a differentiable mapping of $M$ onto $N$. The vector field $X = \partial/\partial x$ on $M$ is not projectable although (1) is satisfied.

5. Deduce from §3.1 that diffeomorphic manifolds have the same dimension.

7. Let $M$ be a manifold, $p \in M$, and $X$ a vector field on $M$ such that $X_p \neq 0$. Then there exists a local chart $\{x_1, \ldots, x_m\}$ on a neighborhood $U$ of $p$ such that $X = \partial/\partial x_1$ on $U$. Deduce that the differential equation $Xu = f (f \in C^\infty(M))$ has a solution $u$ in a neighborhood of $p$.  

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8. Let $M$ be a manifold and $X$, $Y$ two vector fields both $\neq 0$ at a point $o \in M$. For $p$ close to $o$ and $s, t \in \mathbb{R}$ sufficiently small let $\varphi_s(p)$ and $\psi_t(p)$ denote the integral curves through $p$ of $X$ and $Y$, respectively. Let

$$\gamma(t) = \varphi_{-\sqrt{t}}(\varphi_{-\sqrt{t}}(\varphi_{\sqrt{t}}(\varphi_{\sqrt{t}}(o)))).$$

Prove that

$$[X, Y]_o = \lim_{t \to 0} \gamma(t).$$

(Hint: The curves $t \to \varphi_t(\varphi_s(p))$ and $t \to \varphi_t(\varphi_s(p))$ must coincide; deduce $(X^*f)(p) = [d^X/dt^Yf(\varphi_t \cdot p)]_{t=0}$).

**B. The Lie Derivative and the Interior Product**

1. Let $M$ be a manifold, $X$ a vector field on $M$. The Lie derivative $\theta(X): Y \to [X, Y]$ which maps $\mathcal{D}(M)$ into itself can be extended uniquely to a mapping of $\mathcal{D}(M)$ into itself such that:

(i) $\theta(X)f = Xf$ for $f \in C^\infty(M)$.

(ii) $\theta(X)$ is a derivation of $\mathcal{D}(M)$ preserving type of tensors.

(iii) $\theta(X)$ commutes with contractions.

2. Let $\Phi$ be a diffeomorphism of a manifold $M$ onto itself. Then $\Phi$ induces a unique type-preserving automorphism $T \to \Phi \cdot T$ of the tensor algebra $\mathcal{D}(M)$ such that:

(i) The automorphism commutes with contractions.

(ii) $\Phi \cdot X = X^*$, $(X \in \mathcal{D}(M))$, $\Phi \cdot f = f^*$, $(f \in C^\infty(M))$.

Prove that $\Phi \cdot \omega = (\Phi^{-1})^* \omega$ for $\omega \in \mathcal{D}(M)$.

3. Let $g_t$ be a one-parameter Lie transformation group of $M$ and denote by $X$ the vector field on $M$ induced by $g_t$ (Chapter II, §3). Then

$$\theta(X)T = \lim_{t \to 0} \frac{1}{t}(T - g_t \cdot T)$$

for each tensor field $T$ on $M$ ($g_t \cdot T$ is defined in Exercise 2).

4. The Lie derivative $\theta(X)$ on a manifold $M$ has the following properties:

(i) $\theta([X, Y]) = \theta(X) \theta(Y) - \theta(Y) \theta(X)$, $X, Y \in \mathcal{D}(M)$.

(ii) $\theta(X)$ commutes with the alternation $A: \mathcal{D}(M) \to \mathcal{D}(M)$ and therefore induces a derivation of the Grassmann algebra of $M$.

(iii) $\theta(X)d = d\theta(X)$, that is, $\theta(X)$ commutes with exterior differentiation.
5. For $X \in \mathfrak{X}(M)$ there is a unique linear mapping $i(X) : \Omega^k(M) \to \Omega^{k-1}(M)$, the interior product, satisfying:

(i) $i(X)f = 0$ for $f \in \mathcal{C}^\infty(M)$.

(ii) $i(X)\omega = \omega(X)$ for $\omega \in \Omega^1(M)$.

(iii) $i(X) : \Omega^k(M) \to \Omega^{k-1}(M)$ and

\[
i(X) (\omega_1 \wedge \omega_2) = i(X) (\omega_1) \wedge \omega_2 + (-1)^r \omega_1 \wedge i(X) (\omega_2)
\]

if $\omega_1 \in \Omega^r(M)$, $\omega_2 \in \Omega^s(M)$.

6. (cf. H. Cartan [1]). Prove that if $X, Y \in \mathfrak{X}(M)$, $\omega_1, ..., \omega_r \in \Omega^1(M)$,

(i) $i(X)^2 = 0$.

(ii) $i(X)(\omega_1 \wedge ... \wedge \omega_r) = \sum_{1 \leq i < r} (-1)^{i+1} \omega_1(X) \omega_i \wedge ... \wedge \hat{\omega}_i \wedge ... \wedge \omega_r$

where $\omega_i \in \Omega^1(M)$.

(iii) $i([X, Y]) = \theta(X) i(Y) - i(Y) \theta(X)$.

(iv) $\theta(X) = i(X) d + d i(X)$.

C. Affine Connections

2. Let $\bigtriangledown$ be the affine connection on $\mathbb{R}^n$ determined by $\nabla_x(Y) = 0$ for $X = \partial/\partial x_i$, $Y = \partial/\partial x_j$, $1 \leq i, j \leq n$. Find the corresponding affine transformations.

4. Let $M$ be a manifold with a torsion-free affine connection $\bigtriangledown$. Suppose $X_1, ..., X_m$ is a basis for the vector fields on an open subset $U$ of $M$. Let the forms $\omega_1, ..., \omega_m$ on $U$ be determined by $\omega_i(X_j) = \delta_{ij}$.

Prove the formula

\[
d\theta = \sum_{i=1}^m \omega^i \wedge \nabla_x(\theta)
\]

for each differential form $\theta$ on $U$.

5. Let $S$ be a surface in $\mathbb{R}^3$, $X$ and $Y$ two vector fields on $S$. Let $s \in S$, $X_s \neq 0$ and $t \to \gamma(t)$ a curve on $S$ through $s$ such that $\gamma(t) = X_{\gamma(t)}$, $\gamma(0) = s$. Viewing $Y_{\gamma(t)}$ as a vector in $\mathbb{R}^3$ and letting $\pi_s : \mathbb{R}^3 \to S_s$ denote the orthogonal projection put

\[
\nabla_s(Y)_s = \pi_s(\lim_{t \to 0} \frac{1}{t} (Y_{\gamma(t)} - Y_s)).
\]

Prove that this defines an affine connection on $S$. 

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D. Submanifolds

1. Let $M$ and $N$ be differentiable manifolds and $\Phi$ a differentiable mapping of $M$ into $N$. Consider the mapping $\varphi : m \rightarrow (m, \Phi(m)) (m \in M)$ and the graph

$$G_\Phi = \{(m, \Phi(m)) : m \in M\}$$

of $\Phi$ with the topology induced by the product space $M \times N$. Then $\varphi$ is a homeomorphism of $M$ onto $G_\Phi$ and if the differentiable structure of $M$ is transferred to $G_\Phi$ by $\varphi$, the graph $G_\Phi$ becomes a closed submanifold of $M \times N$.

2. Let $N$ be a manifold and $M$ a topological space, $M \subset N$ (as sets).

Show that there exists at most one differentiable structure on the topological space $M$ such that $M$ is a submanifold of $N$.

3. Using the figure 8 as a subset of $\mathbb{R}^2$ show that

(i) A closed connected submanifold of a connected manifold does not necessarily carry the relative topology.

(ii) A subset $M$ of a connected manifold $N$ may have two different topologies and differentiable structures such that in both cases $M$ is a submanifold of $N$.

4. Let $M$ be a submanifold of a manifold $N$ and suppose $M = N$ (as sets). Assuming $M$ to have a countable basis for the open sets, prove that $M = N$ (as manifolds). (Use Prop. 3.2 and Lemma 3.1, Chapter II.)
1. Let $D$ be the open disk $|x| < 1$ in $\mathbb{R}^2$ with the usual differentiable structure but given the Riemannian structure

$$g(u, v) = \frac{(u, v)}{(1 - |x|^2)^2} \quad (u, v \in D)$$

$(u, v)$ denoting the usual inner product on $\mathbb{R}^2$.

(i) Show that the angle between $u$ and $v$ in the Riemannian structure $g$ coincides with the Euclidean angle.

(ii) Show that the Riemannian structure can be written

$$g = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2} \quad (x = x + iy).$$

(iii) Show that the arc length $L$ satisfies

$$L(\gamma_0) \leq L(\gamma)$$

if $\gamma$ is any curve joining the origin $0$ and $x$ ($0 < x < 1$) and $\gamma(t) = tx$ ($0 \leq t \leq 1$).

(iv) Show that the transformation

$$\varphi : z \mapsto \frac{ax + b}{\overline{bx + a}} \quad (|a|^2 - |b|^2 = 1)$$

is an isometry of $D$.

(v) Deduce from (iii) and (iv) that the geodesics in $D$ are the circular arcs perpendicular to the boundary $|x| = 1$.

(vi) Prove from (iii) that

$$d(0, x) = \frac{1}{2} \log \frac{1 + |x|}{1 - |x|} \quad (x \in D)$$

and using (iv) that

$$d(x_1, x_2) = \frac{1}{2} \log \left( \frac{x_1 - b_1}{x_1 - b_1} : \frac{x_2 - b_2}{x_2 - b_2} \right) \quad (x_1, x_2 \in D)$$

with $b_1$ and $b_2$ as in the figure.

(vii) Show that the maps $\varphi$ in (iv) together with the complex conjugation $z \mapsto \overline{z}$ generate the group of all isometries of $D$. 

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