1.13. Exactness of the tensor product.

Proposition 1.13.1. (see [BaKi, 2.1.8]) Let $C$ be a multitensor category. Then the bifunctor $\otimes : C \times C \to C$ is exact in both factors (i.e., biexact).

Proof. The proposition follows from the fact that by Proposition 1.10.9, the functors $V \otimes$ and $\otimes V$ have left and right adjoint functors (the functors of tensoring with the corresponding duals), and any functor between abelian categories which has a left and a right adjoint functor is exact. \qed

Remark 1.13.2. The proof of Proposition 1.13.1 shows that the biadditivity of the functor $\otimes$ holds automatically in any rigid monoidal abelian category. However, this is not the case for bilinearity of $\otimes$, and thus condition of bilinearity of tensor product in the definition of a multitensor category is not redundant.

This may be illustrated by the following example. Let $C$ be the category of finite dimensional $C$-bimodules in which the left and right actions of $\mathbb{R}$ coincide. This category is $C$-linear abelian; namely, it is semisimple with two simple objects $C_+ = 1$ and $C_-$, both equal to $C$ as a real vector space, with bimodule structures $(a, b)z = azb$ and $(a, b)z = azb$, respectively. It is also also rigid monoidal, with $\otimes$ being the tensor product of bimodules. But the tensor product functor is not $C$-bilinear on morphisms (it is only $\mathbb{R}$-bilinear).

Definition 1.13.3. A multiring category over $k$ is a locally finite $k$-linear abelian monoidal category $C$ with biexact tensor product. If in addition $\text{End}(1) = k$, we will call $C$ a ring category.

Thus, the difference between this definition and the definition of a (multi)tensor category is that we don’t require the existence of duals, but instead require the biexactness of the tensor product. Note that Proposition 1.13.1 implies that any multitensor category is a multiring category, and any tensor category is a ring category.

Corollary 1.13.4. For any pair of morphisms $f_1, f_2$ in a multiring category $C$ one has $\text{Im}(f_1 \otimes f_2) = \text{Im}(f_1) \otimes \text{Im}(f_2)$.

Proof. Let $I_1, I_2$ be the images of $f_1, f_2$. Then the morphisms $f_i : X_i \to Y_i$, $i = 1, 2$, have decompositions $X_i \to I_i \to Y_i$, where the sequences

$$X_i \to I_i \to 0, \ 0 \to I_i \to Y_i$$

are exact. Tensoring the sequence $X_1 \to I_1 \to 0$ with $I_2$, by Proposition 1.13.1, we get the exact sequence

$$X_1 \otimes I_2 \to I_1 \otimes I_2 \to 0$$
Tenosring $X_1$ with the sequence $X_2 \to I_2 \to 0$, we get the exact sequence

$$X_1 \otimes X_2 \to X_1 \otimes I_2 \to 0.$$ Combining these, we get an exact sequence

$$X_1 \otimes X_2 \to I_1 \otimes I_2 \to 0.$$ Arguing similarly, we show that the sequence

$$0 \to I_1 \otimes I_2 \to Y_1 \otimes Y_2$$ is exact. This implies the statement.

**Proposition 1.13.5.** If $\mathcal{C}$ is a multiring category with right duals, then the right dualization functor is exact. The same applies to left duals.

**Proof.** Let $0 \to X \to Y \to Z \to 0$ be an exact sequence. We need to show that the sequence $0 \to Z^* \to Y^* \to X^* \to 0$ is exact. Let $T$ be any object of $\mathcal{C}$, and consider the sequence

$$0 \to \text{Hom}(T, Z^*) \to \text{Hom}(T, Y^*) \to \text{Hom}(T, X^*).$$

By Proposition 1.10.9, it can be written as

$$0 \to \text{Hom}(T \otimes Z, 1) \to \text{Hom}(T \otimes Y, 1) \to \text{Hom}(T \otimes X, 1),$$

which is exact, since the sequence

$$T \otimes X \to T \otimes Y \to T \otimes Z \to 0$$
is exact, by the exactness of the functor $T \otimes$. This implies that the sequence $0 \to Z^* \to Y^* \to X^* \to 0$ is exact.

Similarly, consider the sequence

$$0 \to \text{Hom}(X^*, T) \to \text{Hom}(Y^*, T) \to \text{Hom}(Z^*, T).$$

By Proposition 1.10.9, it can be written as

$$0 \to \text{Hom}(1, X \otimes T) \to \text{Hom}(1, Y \otimes T) \to \text{Hom}(1, Z \otimes T),$$

which is exact since the sequence

$$0 \to X \otimes T \to Y \otimes T \to Z \otimes T$$
is exact, by the exactness of the functor $\otimes T$. This implies that the sequence $Z^* \to Y^* \to X^* \to 0$ is exact.

**Proposition 1.13.6.** Let $P$ be a projective object in a multiring category $\mathcal{C}$. If $X \in \mathcal{C}$ has a right dual, then the object $P \otimes X$ is projective. Similarly, if $X \in \mathcal{C}$ has a left dual, then the object $X \otimes P$ is projective.
Proof. In the first case by Proposition 1.10.9 we have $\text{Hom}(P \otimes X, Y) = \text{Hom}(P, Y \otimes X^*)$, which is an exact functor of $Y$, since the functors $\otimes X^*$ and $\text{Hom}(P, \bullet)$ are exact. So $P \otimes X$ is projective. The second case is similar.

**Corollary 1.13.7.** If $\mathcal{C}$ multiring category with right duals, then $1 \in \mathcal{C}$ is a projective object if and only if $\mathcal{C}$ is semisimple.

*Proof.* If $1$ is projective then for any $X \in \mathcal{C}$, $X \cong 1 \otimes X$ is projective. This implies that $\mathcal{C}$ is semisimple. The converse is obvious.

1.14. **Quasi-tensor and tensor functors.**

**Definition 1.14.1.** Let $\mathcal{C}$, $\mathcal{D}$ be multiring categories over $k$, and $F : \mathcal{C} \to \mathcal{D}$ be an exact and faithful functor.

(i) $F$ is said to be a quasi-tensor functor if it is equipped with a functorial isomorphism $J : F(\bullet) \otimes F(\bullet) \to F(\bullet \otimes \bullet)$, and $F(1) = 1$.

(ii) A quasi-tensor functor $(F, J)$ is said to be a tensor functor if $J$ is a monoidal structure (i.e., satisfies the monoidal structure axiom).

**Example 1.14.2.** The functors of Examples 1.6.1,1.6.2 and Subsection 1.7 (for the categories $\text{Vec}_G^\omega$) are tensor functors. The identity functor $\text{Vec}_G^\omega \to \text{Vec}_G^{\omega^2}$ for non-cohomologous 3-cocycles $\omega_1, \omega_2$ is not a tensor functor, but it can be made quasi-tensor by any choice of $J$.

1.15. **Semisimplicity of the unit object.**

**Theorem 1.15.1.** In any multiring category, $\text{End}(1)$ is a semisimple algebra, so it is isomorphic to a direct sum of finitely many copies of $k$.

*Proof.* By Proposition 1.2.7, $\text{End}(1)$ is a commutative algebra, so it is sufficient to show that for any $a \in \text{End}(1)$ such that $a^2 = 0$ we have $a = 0$. Let $J = \text{Im}(a)$. Then by Corollary 1.13.4 $J \otimes J = \text{Im}(a \otimes a) = \text{Im}(a^2 \otimes 1) = 0$.

Now let $K = \text{Ker}(a)$. Then by Corollary 1.13.4, $K \otimes J$ is the image of $1 \otimes a$ on $K \otimes 1$. But since $K \otimes 1$ is a subobject of $1 \otimes 1$, this is the same as the image of $a \otimes 1$ on $K \otimes 1$, which is zero. So $K \otimes J = 0$.

Now tensoring the exact sequence $0 \to K \to 1 \to J \to 0$ with $J$, and applying Proposition 1.13.1, we get that $J = 0$, so $a = 0$.

Let $\{p_i\}_{i \in I}$ be the primitive idempotents of the algebra $\text{End}(1)$. Let $1_i$ be the image of $p_i$. Then we have $1 = \bigoplus_{i \in I} 1_i$.

**Corollary 1.15.2.** In any multiring category $\mathcal{C}$ the unit object $1$ is isomorphic to a direct sum of pairwise non-isomorphic indecomposable objects: $1 \cong \bigoplus_{i \in I} 1_i$. 
Exercise 1.15.3. One has \( 1_i \otimes 1_j = 0 \) for \( i \neq j \). There are canonical isomorphisms \( 1_i \otimes 1_i \cong 1_i \), and \( 1_i \cong 1_i^* \).

Let \( C_{ij} := 1_i \otimes C \otimes 1_j \).

Definition 1.15.4. The subcategories \( C_{ij} \) will be called the component subcategories of \( C \).

Proposition 1.15.5. Let \( C \) be a multiring category.

1. \( C = \bigoplus_{i,j \in I} C_{ij} \). Thus every indecomposable object of \( C \) belongs to some \( C_{ij} \).
2. The tensor product maps \( C_{ij} \times C_{kl} \) to \( C_{il} \), and it is zero unless \( j = k \).
3. The categories \( C_{ii} \) are ring categories with unit objects \( 1_i \) (which are tensor categories if \( C \) is rigid).
4. The functors of left and right duals, if they are defined, map \( C_{ij} \) to \( C_{ji} \).

Exercise 1.15.6. Prove Proposition 1.15.5.

Proposition 1.15.5 motivates the terms “multiring category” and “multitensor category”, as such a category gives us multiple ring categories, respectively tensor categories \( C_{ii} \).

Remark 1.15.7. Thus, a multiring category may be considered as a 2-category with objects being elements of \( I \), 1-morphisms from \( j \) to \( i \) forming the category \( C_{ij} \), and 2-morphisms being 1-morphisms in \( C \).

Theorem 1.15.8. (i) In a ring category with right duals, the unit object \( 1 \) is simple.
(ii) In a multiring category with right duals, the unit object \( 1 \) is semisimple, and is a direct sum of pairwise non-isomorphic simple objects \( 1_i \).

Proof. Clearly, (i) implies (ii) (by applying (i) to the component categories \( C_{ii} \)). So it is enough to prove (i).

Let \( X \) be a simple subobject of \( 1 \) (it exists, since \( 1 \) has finite length). Let

\[(1.15.1) \quad 0 \rightarrow X \rightarrow 1 \rightarrow Y \rightarrow 0\]

be the corresponding exact sequence. By Proposition 1.13.5, the right dualization functor is exact, so we get an exact sequence

\[(1.15.2) \quad 0 \rightarrow Y^* \rightarrow 1 \rightarrow X^* \rightarrow 0.\]

Tensoring this sequence with \( X \) on the left, we obtain

\[(1.15.3) \quad 0 \rightarrow X \otimes Y^* \rightarrow X \rightarrow X \otimes X^* \rightarrow 0,\]
Since $X$ is simple and $X \otimes X^* \neq 0$ (because the coevaluation morphism is nonzero) we obtain that $X \otimes X^* \cong X$. So we have a surjective composition morphism $1 \to X \otimes X^* \to X$. From this and (1.15.1) we have a nonzero composition morphism $1 \to X \leftrightarrow 1$. Since $\text{End}(1) = k$, this morphism is a nonzero scalar, whence $X = 1$.  

**Corollary 1.15.9.** In a ring category with right duals, the evaluation morphisms are surjective and the coevaluation morphisms are injective.

**Exercise 1.15.10.** Let $C$ be a multiring category with right duals, and $X \in C_{ij}$ and $Y \in C_{jk}$ be nonzero.

(a) Show that $X \otimes Y \neq 0$.

(b) Deduce that $\text{length}(X \otimes Y) \geq \text{length}(X)\text{length}(Y)$.

(c) Show that if $C$ is a ring category with right duals then an invertible object in $C$ is simple.

(d) Let $X$ be an object in a multiring category with right duals such that $X \otimes X^* \cong 1$. Show that $X$ is invertible.

**Example 1.15.11.** An example of a ring category where the unit object is not simple is the category $C$ of finite dimensional representations of the quiver of type $A_2$. Such representations are triples $(V, W, A)$, where $V, W$ are finite dimensional vector spaces, and $A : V \to W$ is a linear operator. The tensor product on such triples is defined by the formula

$$(V, W, A) \otimes (V', W', A') = (V \otimes V', W \otimes W', A \otimes A'),$$

with obvious associativity isomorphisms, and the unit object $(k, k, \text{Id})$. Of course, this category has neither right nor left duals.

1.16. **Grothendieck rings.** Let $C$ be a locally finite abelian category over $k$. If $X$ and $Y$ are objects in $C$ such that $Y$ is simple then we denote by $[X:Y]$ the multiplicity of $Y$ in the Jordan-Hölder composition series of $X$.

Recall that the Grothendieck group $\text{Gr}(C)$ is the free abelian group generated by isomorphism classes $X_i$, $i \in I$ of simple objects in $C$, and that to every object $X$ in $C$ we can canonically associate its class $[X] \in \text{Gr}(C)$ given by the formula $[X] = \sum_i [X_i]X_i$. It is obvious that if

$$0 \to X \to Y \to Z \to 0$$

then $[Y] = [X] + [Z]$. When no confusion is possible, we will write $X$ instead of $[X]$.
Now let $\mathcal{C}$ be a multiring category. The tensor product on $\mathcal{C}$ induces a natural multiplication on $\text{Gr}(\mathcal{C})$ defined by the formula

$$X_i X_j := [X_i \otimes X_j] = \sum_{k \in I} [X_i \otimes X_j : X_k] X_k.$$

**Lemma 1.16.1.** The above multiplication on $\text{Gr}(\mathcal{C})$ is associative.

**Proof.** Since the tensor product functor is exact,

$$[(X_i \otimes X_j) \otimes X_p : X_l] = \sum_{k} [X_i \otimes X_j : X_k] [X_k \otimes X_p : X_l].$$

On the other hand,

$$[X_i \otimes (X_j \otimes X_p) : X_l] = \sum_{k} [X_j \otimes X_p : X_k] [X_i \otimes X_k : X_l].$$

Thus the associativity of the multiplication follows from the isomorphism $(X_i \otimes X_j) \otimes X_p \cong X_i \otimes (X_j \otimes X_p).$  

Thus $\text{Gr}(\mathcal{C})$ is an associative ring with the unit 1. It is called the Grothendieck ring of $\mathcal{C}$.

The following proposition is obvious.

**Proposition 1.16.2.** Let $\mathcal{C}$ and $\mathcal{D}$ be multiring categories and $F : \mathcal{C} \rightarrow \mathcal{D}$ be a quasi-tensor functor. Then $F$ defines a homomorphism of unital rings $[F] : \text{Gr}(\mathcal{C}) \rightarrow \text{Gr}(\mathcal{D})$.

Thus, we see that (multi)ring categories categorify rings (which justifies the terminology), while quasi-tensor (in particular, tensor) functors between them categorify unital ring homomorphisms. Note that Proposition 1.15.5 may be regarded as a categorical analog of the Peirce decomposition in classical algebra.

1.17. **Groupoids.** The most basic examples of multitensor categories arise from finite groupoids. Recall that a groupoid is a small category where all morphisms are isomorphisms. Thus a groupoid $\mathcal{G}$ entails a set $X$ of objects of $\mathcal{G}$ and a set $G$ of morphisms of $\mathcal{G}$, the source and target maps $s, t : G \rightarrow X$, the composition map $\mu : G \times_X G \rightarrow G$ (where the fibered product is defined using $s$ in the first factor and using $t$ in the second factor), the unit morphism map $u : X \rightarrow G$, and the inversion map $i : G \rightarrow G$ satisfying certain natural axioms, see e.g. [Ren] for more details.

Here are some examples of groupoids.

(1) Any group $G$ is a groupoid $\mathcal{G}$ with a single object whose set of morphisms to itself is $G$. 

(2) Let $X$ be a set and let $G = X \times X$. Then the product groupoid $\mathcal{G}(X) := (X, G)$ is a groupoid in which $s$ is the first projection, $t$ is the second projection, $u$ is the diagonal map, and $i$ is the permutation of factors. In this groupoid for any $x, y \in X$ there is a unique morphism from $x$ to $y$.

(3) A more interesting example is the transformation groupoid $T(G, X)$ arising from the action of a group $G$ on a set $X$. The set of objects of $T(G, X)$ is $X$, and arrows correspond to triples $(g, x, y)$ where $y = gx$ with an obvious composition law. In other words, the set of morphisms is $G \times X$ and $s(g, x) = x$, $t(g, x) = gx$, $u(x) = (1, x)$, $i(g, x) = (g^{-1}, gx)$.

Let $\mathcal{G} = (X, G, \mu, s, t, u, i)$ be a finite groupoid (i.e., $G$ is finite) and let $\mathcal{C}(\mathcal{G})$ be the category of finite dimensional vector spaces graded by the set $G$ of morphisms of $\mathcal{G}$, i.e., vector spaces of the form $V = \bigoplus_{g \in G} V_g$. Introduce a tensor product on $\mathcal{C}(\mathcal{G})$ by the formula

$$(1.17.1) \quad (V \otimes W)_g = \bigoplus_{(g_1, g_2) : g_1 g_2 = g} V_{g_1} \otimes W_{g_2}.$$ 

Then $\mathcal{C}(\mathcal{G})$ is a multitensor category. The unit object is $1 = \bigoplus_{x \in X} 1_x$, where $1_x$ is a 1-dimensional vector space which sits in degree $\text{id}_x$ in $G$. The left and right duals are defined by $(V^*)_g = (V)_g = V_{g^{-1}}$.

We invite the reader to check that the component subcategories $\mathcal{C}(\mathcal{G})_{xy}$ are the categories of vector spaces graded by $\text{Mor}(y, x)$.

We see that $\mathcal{C}(\mathcal{G})$ is a tensor category if and only if $\mathcal{G}$ is a group, which is the case of $\text{Vec}_G$ already considered in Example 1.3.6. Note also that if $X = \{1, ..., n\}$ then $\mathcal{C}(\mathcal{G}(X))$ is naturally equivalent to $M_n(\text{Vec})$.

**Exercise 1.17.1.** Let $C_i$ be isomorphism classes of objects in a finite groupoid $\mathcal{G}$, $n_i = |C_i|$, $x_i \in C_i$ be representatives of $C_i$, and $G_i = \text{Aut}(x_i)$ be the corresponding automorphism groups. Show that $\mathcal{C}(\mathcal{G})$ is (non-canonically) monoidally equivalent to $\bigoplus_{i} M_{n_i}(\text{Vec}_{G_i})$.

**Remark 1.17.2.** The finite length condition in Definition 1.12.3 is not superfluous: there exists a rigid monoidal $k$-linear abelian category with bilinear tensor product which contains objects of infinite length. An example of such a category is the category $\mathcal{C}$ of Jacobi matrices of finite dimensional vector spaces. Namely, the objects of $\mathcal{C}$ are semi-infinite matrices $V = \{V_{ij}\}_{i \in \mathbb{Z}^+}$ of finite dimensional vector spaces $V_{ij}$ with finitely many non-zero diagonals, and morphisms are matrices of linear maps. The tensor product in this category is defined by the
formula
(1.17.2) \[(V \otimes W)_t = \sum_j V_{ij} \otimes W_{jt},\]
and the unit object \(1\) is defined by the condition \(1_{ij} = k^{\delta_{ij}}\). The left and right duality functors coincide and are given by the formula
(1.17.3) \[(V^*)_{ij} = (V_{ji})^*.\]

The evaluation map is the direct sum of the canonical maps \(V^*_{ij} \otimes V_{ij} \to 1_{jj}\), and the coevaluation map is a direct sum of the canonical maps \(1_{ii} \to V_{ij} \otimes V^*_{ij}\).

Note that the category \(C\) is a subcategory of the category \(C'\) of \(\mathbb{G}(\mathbb{Z}_+)\)-graded vector spaces with finite dimensional homogeneous components. Note also that the category \(C'\) is not closed under the tensor product defined by (1.17.2) but the category \(C\) is.

**Exercise 1.17.3.** (1) Show that if \(X\) is a finite set then the group of invertible objects of the category \(C(\mathbb{G}(X))\) is isomorphic to \(\text{Aut}(X)\).

(2) Let \(C\) be the category of Jacobi matrices of vector spaces from Example 1.17.2. Show that the statement Exercise 1.15.10(d) fails for \(C\). Thus the finite length condition is important in Exercise 1.15.10.

1.18. **Finite abelian categories and exact faithful functors.**

**Definition 1.18.1.** A \(k\)-linear abelian category \(C\) is said to be finite if it is equivalent to the category \(A - \text{mod}\) of finite dimensional modules over a finite dimensional \(k\)-algebra \(A\).

Of course, the algebra \(A\) is not canonically attached to the category \(C\); rather, \(C\) determines the Morita equivalence class of \(A\). For this reason, it is often better to use the following “intrinsic” definition, which is well known to be equivalent to Definition 1.18.1:

**Definition 1.18.2.** A \(k\)-linear abelian category \(C\) is finite if
(i) \(C\) has finite dimensional spaces of morphisms;
(ii) every object of \(C\) has finite length;
(iii) \(C\) has enough projectives, i.e., every simple object of \(C\) has a projective cover; and
(iv) there are finitely many isomorphism classes of simple objects.

Note that the first two conditions are the requirement that \(C\) be locally finite.

Indeed, it is clear that if \(A\) is a finite dimensional algebra then \(A - \text{mod}\) clearly satisfies (i)-(iv), and conversely, if \(C\) satisfies (i)-(iv), then
one can take $A = \text{End}(P)^{op}$, where $P$ is a projective generator of $\mathcal{C}$ (e.g., $P = \bigoplus_{i=1}^{n} P_i$, where $P_i$ are projective covers of all the simple objects $X_i$).

A projective generator $P$ of $\mathcal{C}$ represents a functor $F = F_P : \mathcal{C} \to \text{Vec}$ from $\mathcal{C}$ to the category of finite dimensional $k$-vector spaces, given by the formula $F(X) = \text{Hom}(P, X)$. The condition that $P$ is projective translates into the exactness property of $F$, and the condition that $P$ is a generator (i.e., covers any simple object) translates into the property that $F$ is faithful (does not kill nonzero objects or morphisms). Moreover, the algebra $A = \text{End}(P)^{op}$ can be alternatively defined as $\text{End}(F)$, the algebra of functorial endomorphisms of $F$. Conversely, it is well known (and easy to show) that any exact faithful functor $F : \mathcal{C} \to \text{Vec}$ is represented by a unique (up to a unique isomorphism) projective generator $P$.

Now let $\mathcal{C}$ be a finite $k$-linear abelian category, and $F_1, F_2 : \mathcal{C} \to \text{Vec}$ be two exact faithful functors. Define the functor $F_1 \otimes F_2 : \mathcal{C} \times \mathcal{C} \to \text{Vec}$ by $(F_1 \otimes F_2)(X, Y) := F_1(X) \otimes F_2(Y)$.

**Proposition 1.18.3.** There is a canonical algebra isomorphism $\alpha_{F_1, F_2} : \text{End}(F_1) \otimes \text{End}(F_2) \cong \text{End}(F_1 \otimes F_2)$ given by

$$
\alpha_{F_1, F_2}(\eta_1 \otimes \eta_2)|_{F_1(X) \otimes F_2(Y)} := \eta_1|_{F_1(X)} \otimes \eta_2|_{F_2(Y)},
$$

where $\eta_i \in \text{End}(F_i), i = 1, 2$.

**Exercise 1.18.4.** Prove Proposition 1.18.3.

### 1.19. Fiber functors.

Let $\mathcal{C}$ be a $k$-linear abelian monoidal category.

**Definition 1.19.1.** A quasi-fiber functor on $\mathcal{C}$ is an exact faithful functor $F : \mathcal{C} \to \text{Vec}$ from $\mathcal{C}$ to the category of finite dimensional $k$-vector spaces, such that $F(1) = k$, equipped with an isomorphism $J : F(\bullet) \otimes F(\bullet) \to F(\bullet \otimes \bullet)$. If in addition $J$ is a monoidal structure (i.e. satisfies the monoidal structure axiom), one says that $F$ is a fiber functor.

**Example 1.19.2.** The forgetful functors $\text{Vec}_G \to \text{Vec}$, $\text{Rep}(G) \to \text{Vec}$ are naturally fiber functors, while the forgetful functor $\text{Vec}_G^{\omega} \to \text{Vec}$ is quasi-fiber, for any choice of the isomorphism $J$ (we have seen that if $\omega$ is cohomologically nontrivial, then $\text{Vec}_G^{\omega}$ does not admit a fiber functor). Also, the functor $\text{Loc}(X) \to \text{Vec}$ on the category of local systems on a connected topological space $X$ which attaches to a local system $E$ its fiber $E_x$ at a point $x \in X$ is a fiber functor, which justifies the terminology. (Note that if $X$ is Hausdorff, then this functor can be identified with the abovementioned forgetful functor $\text{Rep}(\pi_1(X, x)) \to \text{Vec}$).
Exercise 1.19.3. Show that if an abelian monoidal category $C$ admits a quasi-fiber functor, then it is a ring category, in which the object $1$ is simple. So if in addition $C$ is rigid, then it is a tensor category.

1.20. Coalgebras.

Definition 1.20.1. A coalgebra (with counit) over a field $k$ is a $k$-vector space $C$ together with a comultiplication (or coproduct) $\Delta : C \to C \otimes C$ and counit $\varepsilon : C \to k$ such that

(i) $\Delta$ is coassociative, i.e.,

$$(\Delta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \Delta) \circ \Delta$$

as maps $C \to C^{\otimes 3}$;

(ii) one has

$$(\varepsilon \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \varepsilon) \circ \Delta = \text{Id}$$

as maps $C \to C$ (the “counit axiom”).

Definition 1.20.2. A left comodule over a coalgebra $C$ is a vector space $M$ together with a linear map $\pi : M \to C \otimes M$ (called the coaction map), such that for any $m \in M$, one has

$$(\Delta \otimes \text{Id})(\pi(m)) = (\text{Id} \otimes \pi)(\pi(m)), \quad (\varepsilon \otimes \text{Id})(\pi(m)) = m.$$  

Similarly, a right comodule over $C$ is a vector space $M$ together with a linear map $\pi : M \to M \otimes C$, such that for any $m \in M$, one has

$$(\pi \otimes \text{Id})(\pi(m)) = (\text{Id} \otimes \Delta)(\pi(m)), \quad (\text{Id} \otimes \varepsilon)(\pi(m)) = m.$$  

For example, $C$ is a left and right comodule with $\pi = \Delta$, and so is $k$, with $\pi = \varepsilon$.

Exercise 1.20.3. (i) Show that if $C$ is a coalgebra then $C^*$ is an algebra, and if $A$ is a finite dimensional algebra then $A^*$ is a coalgebra.

(ii) Show that for any coalgebra $C$, any (left or right) $C$-comodule $M$ is a (respectively, right or left) $C^*$-module, and the converse is true if $C$ is finite dimensional.

Exercise 1.20.4. (i) Show that any coalgebra $C$ is a sum of finite dimensional subcoalgebras.

Hint. Let $c \in C$, and let

$$(\Delta \otimes \text{Id}) \circ \Delta(c) = (\text{Id} \otimes \Delta) \circ \Delta(c) = \sum_i c_i^1 \otimes c_i^2 \otimes c_i^3.$$  

Show that $\text{span}(c_i^2)$ is a subcoalgebra of $C$ containing $c$.

(ii) Show that any $C$-comodule is a sum of finite dimensional subcomodules.
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