1.45. **Tensor categories with finitely many simple objects. Frobenius-Perron dimensions.** Let $A$ be a $\mathbb{Z}_+$-ring with $\mathbb{Z}_+$-basis $I$. 

**Definition 1.45.1.** We will say that $A$ is *transitive* if for any $X, Z \in I$ there exist $Y_1, Y_2 \in I$ such that $XY_1$ and $Y_2X$ involve $Z$ with a nonzero coefficient.

**Proposition 1.45.2.** If $\mathcal{C}$ is a ring category with right duals then $\text{Gr}(\mathcal{C})$ is a transitive unital $\mathbb{Z}_+$-ring.

**Proof.** Recall from Theorem 1.15.8 that the unit object 1 in $\mathcal{C}$ is simple. So $\text{Gr}(\mathcal{C})$ is unital. This implies that for any simple objects $X, Z$ of $\mathcal{C}$, the object $X \otimes X^* \otimes Z$ contains $Z$ as a composition factor (as $X \otimes X^*$ contains 1 as a composition factor), so one can find a simple object $Y_1$ occurring in $X^* \otimes Z$ such that $Z$ occurs in $X \otimes Y_1$. Similarly, the object $Z \otimes X^* \otimes X$ contains $Z$ as a composition factor, so one can find a simple object $Y_2$ occurring in $Z \otimes X^*$ such that $Z$ occurs in $Y_2 \otimes X$. Thus $\text{Gr}(\mathcal{C})$ is transitive. \hfill $\square$

Let $A$ be a transitive unital $\mathbb{Z}_+$-ring of finite rank. Define the group homomorphism $\text{FPdim} : A \to \mathbb{C}$ as follows. For $X \in I$, let $\text{FPdim}(X)$ be the maximal nonnegative eigenvalue of the matrix of left multiplication by $X$. It exists by the Frobenius-Perron theorem, since this matrix has nonnegative entries. Let us extend $\text{FPdim}$ from the basis $I$ to $A$ by additivity.

**Definition 1.45.3.** The function $\text{FPdim}$ is called the *Frobenius-Perron dimension*.

In particular, if $\mathcal{C}$ is a ring category with right duals and finitely many simple objects, then we can talk about Frobenius-Perron dimensions of objects of $\mathcal{C}$.

**Proposition 1.45.4.** Let $X \in I$.

1. The number $\alpha = \text{FPdim}(X)$ is an algebraic integer, and for any algebraic conjugate $\alpha'$ of $\alpha$ we have $\alpha \geq |\alpha'|$.

2. $\text{FPdim}(X) \geq 1$.

**Proof.** (1) Note that $\alpha$ is an eigenvalue of the integer matrix $N_X$ of left multiplication by $X$, hence $\alpha$ is an algebraic integer. The number $\alpha'$ is a root of the characteristic polynomial of $N_X$, so it is also an eigenvalue of $N_X$. Thus by the Frobenius-Perron theorem $\alpha \geq |\alpha'|$.

(2) Let $r$ be the number of algebraic conjugates of $\alpha$. Then $\alpha' \geq N(\alpha)$ where $N(\alpha)$ is the norm of $\alpha$. This implies the statement since $N(\alpha) \geq 1$. \hfill $\square$
Proposition 1.45.5.  
(1) The function $\text{FPdim} : A \to \mathbb{C}$ is a ring homomorphism.
(2) There exists a unique, up to scaling, element $R \in A_{\mathbb{C}} := A \otimes_{\mathbb{Z}} \mathbb{C}$ such that $XR = \text{FPdim}(X)R$, for all $X \in A$. After an appropriate normalization this element has positive coefficients, and satisfies $\text{FPdim}(R) > 0$ and $RY = \text{FPdim}(Y)R$, $Y \in A$.
(3) $\text{FPdim}$ is a unique nonzero character of $A$ which takes nonnegative values on $I$.
(4) If $X \in A$ has nonnegative coefficients with respect to the basis of $A$, then $\text{FPdim}(X)$ is the largest nonnegative eigenvalue $\lambda(N_X)$ of the matrix $N_X$ of multiplication by $X$.

Proof. Consider the matrix $M$ of right multiplication by $\sum_{X \in I} X$ in $A$ in the basis $I$. By transitivity, this matrix has strictly positive entries, so by the Frobenius-Perron theorem, part (2), it has a unique, up to scaling, eigenvector $R \in A_{\mathbb{C}}$ with eigenvalue $\lambda(M)$ (the maximal positive eigenvalue of $M$). Furthermore, this eigenvector can be normalized to have strictly positive entries.

Since $R$ is unique, it satisfies the equation $XR = d(X)R$ for some function $d : A \to \mathbb{C}$. Indeed, $XR$ is also an eigenvector of $M$ with eigenvalue $\lambda(M)$, so it must be proportional to $R$. Furthermore, it is clear that $d$ is a character of $A$. Since $R$ has positive entries, $d(X) = \text{FPdim}(X)$ for $X \in I$. This implies (1). We also see that $\text{FPdim}(X) > 0$ for $X \in I$ (as $R$ has strictly positive coefficients), and hence $\text{FPdim}(R) > 0$.

Now, by transitivity, $R$ is the unique, up to scaling, solution of the system of linear equations $XR = \text{FPdim}(X)R$ (as the matrix $N$ of left multiplication by $\sum_{X \in I} X$ also has positive entries). Hence, $RY = d'(Y)R$ for some character $d'$. Applying $\text{FPdim}$ to both sides and using that $\text{FPdim}(R) > 0$, we find $d' = \text{FPdim}$, proving (2).

If $\chi$ is another character of $A$ taking positive values on $I$, then the vector with entries $\chi(Y)$, $Y \in I$ is an eigenvector of the matrix $N$ of the left multiplication by the element $\sum_{X \in I} X$. Because of transitivity of $A$ the matrix $N$ has positive entries. By the Frobenius-Perron theorem there exists a positive number $\lambda$ such that $\chi(Y) = \lambda \text{FPdim}(Y)$. Since $\chi$ is a character, $\lambda = 1$, which completes the proof.

Finally, part (4) follows from part (2) and the Frobenius-Perron theorem (part (3)). $\square$

Example 1.45.6. Let $\mathcal{C}$ be the category of finite dimensional representations of a quasi-Hopf algebra $H$, and $A$ be its Grothendieck ring. Then by Proposition 1.10.9, for any $X, Y \in \mathcal{C}$

$$\dim \text{Hom}(X \otimes H, Y) = \dim \text{Hom}(H, \ast X \otimes Y) = \dim(X) \dim(Y),$$
where $H$ is the regular representation of $H$. Thus $X \otimes H = \dim(X)H$, so $\FPdim(X) = \dim(X)$ for all $X$, and $R = H$ up to scaling.

This example motivates the following definition.

**Definition 1.45.7.** The element $R$ will be called a regular element of $A$.

**Proposition 1.45.8.** Let $A$ be as above and $*: I \to I$ be a bijection which extends to an anti-automorphism of $A$. Then $\FPdim$ is invariant under $*$.

**Proof.** Let $X \in I$. Then the matrix of right multiplication by $X^*$ is the transpose of the matrix of left multiplication by $X$ modified by the permutation $*$. Thus the required statement follows from Proposition 1.45.5(2).

**Corollary 1.45.9.** Let $C$ be a ring category with right duals and finitely many simple objects, and let $X$ be an object in $C$. If $\FPdim(X) = 1$, then $X$ is invertible.

**Proof.** By Exercise 1.15.10(d) it is sufficient to show that $X \otimes X^* = 1$. This follows from the facts that $1$ is contained in $X \otimes X^*$ and $\FPdim(X \otimes X^*) = \FPdim(X) \FPdim(X^*) = 1$.

**Proposition 1.45.10.** Let $f: A_1 \to A_2$ be a unital homomorphism of transitive unital $\mathbb{Z}_+$-rings of finite rank, whose matrix in their $\mathbb{Z}_+$-bases has non-negative entries. Then

1. $f$ preserves Frobenius-Perron dimensions.
2. Let $I_1, I_2$ be the $\mathbb{Z}_+$-bases of $A_1, A_2$, and suppose that for any $Y$ in $I_2$ there exists $X \in I_1$ such that the coefficient of $Y$ in $f(X)$ is non-zero. If $R$ is a regular element of $A_1$ then $f(R)$ is a regular element of $A_2$.

**Proof.** (1) The function $X \mapsto \FPdim(f(X))$ is a nonzero character of $A_1$ with nonnegative values on the basis. By Proposition 1.45.5(3), $\FPdim(f(X)) = \FPdim(X)$ for all $X$ in $I$. (2) By part (1) we have

$$(1.45.1) \quad f(\sum_{X \in I_1} X)f(R_1) = \FPdim(f(\sum_{X \in I_1} X))f(R_1).$$

But $f(\sum_{X \in I_1} X)$ has strictly positive coefficients in $I_2$, hence $f(R_1) = \beta R_2$ for some $\beta > 0$. Applying $\FPdim$ to both sides, we get the result.

**Corollary 1.45.11.** Let $C$ and $D$ be tensor categories with finitely many classes of simple objects. If $F: C \to D$ be a quasi-tensor functor, then $\FPdim_D(F(X)) = \FPdim_C(X)$ for any $X$ in $C$. 
Example 1.45.12. (Tambara-Yamagami fusion rings) Let $G$ be a finite group, and $TY_G$ be an extension of the unital based ring $\mathbb{Z}[G]$:

$$TY_G := \mathbb{Z}[G] \oplus \mathbb{Z}X,$$

where $X$ is a new basis vector with $gX = Xg = X$, $X^2 = \sum_{g \in G} g$. This is a fusion ring, with $X^* = X$. It is easy to see that $\text{FPdim}(g) = 1$, $\text{FPdim}(X) = |G|^{1/2}$. We will see later that these rings are categorifiable if and only if $G$ is abelian.

Example 1.45.13. (Verlinde rings for $\mathfrak{sl}_2$). Let $k$ be a nonnegative integer. Define a unital $\mathbb{Z}_+$-ring $\text{Ver}_k = \text{Ver}_k(\mathfrak{sl}_2)$ with basis $V_i$, $i = 0, ..., k$ ($V_0 = 1$), with duality given by $V_i^* = V_i$ and multiplication given by the truncated Clebsch-Gordan rule:

$$V_i \otimes V_j = \bigoplus_{l = |i-j|, i+j-1 \in \mathbb{Z}} V_l.$$  \hspace{1cm} (1.45.2)

It other words, one computes the product by the usual Clebsch-Gordan rule, and then deletes the terms that are not defined ($V_i$ with $i > k$) and also their mirror images with respect to point $k+1$. We will show later that this ring admits categorifications coming from quantum groups at roots of unity.

Note that $\text{Ver}_0 = \mathbb{Z}$, $\text{Ver}_1 = \mathbb{Z}[\mathbb{Z}_2]$, $\text{Ver}_2 = TY_{\mathbb{Z}_2}$. The latter is called the Ising fusion ring, as it arises in the Ising model of statistical mechanics.

Exercise 1.45.14. Show that $\text{FPdim}(V_j) = [j+1]_q := \frac{q^{j+1} - q^{-j-1}}{q-q^{-1}}$, where $q = e^{2\pi i/3}$.

Note that the Verlinde ring has a subring $\text{Ver}_k^0$ spanned by $V_j$ with even $j$. If $k = 3$, this ring has basis 1, $X = V_2$ with $X^2 = X + 1$, $X^* = X$. This ring is called the Yang-Lee fusion ring. In the Yang-Lee ring, $\text{FPdim}(X)$ is the golden ratio $\frac{1+\sqrt{5}}{2}$.

Note that one can define the generalized Yang-Lee fusion rings $YL_n$ $n \in \mathbb{Z}_+$, with basis 1, $X$, multiplication $X^2 = 1 + nX$ and duality $X^* = X$. It is, however, shown in [O2] that these rings are not categorifiable when $n > 1$.

Proposition 1.45.15. (Kronecker) Let $B$ be a matrix with nonnegative integer entries, such that $\lambda(BB^\top) = \lambda(B)^2$. If $\lambda(B) < 2$ then $\lambda(B) = 2\cos(\pi/n)$ for some integer $n \geq 2$.

Proof. Let $\lambda(B) = q + q^{-1}$. Then $q$ is an algebraic integer, and $|q| = 1$. Moreover, all conjugates of $\lambda(B)^2$ are nonnegative (since they are
eigenvalues of the matrix $BB^T$, which is symmetric and nonnegative definite), so all conjugates of $\lambda(B)$ are real. Thus, if $q_*$ is a conjugate of $q$ then $q_* + q_*^{-1}$ is real with absolute value $< 2$ (by the Frobenius-Perron theorem), so $|q_*| = 1$. By a well known result in elementary algebraic number theory, this implies that $q$ is a root of unity: $q = e^{2\pi ik/m}$, where $k$ and $m$ are coprime. By the Frobenius-Perron theorem, so $k = \pm 1$, and $m$ is even (indeed, if $m = 2p + 1$ is odd then $|q^p + q^{-p}| > |q + q^{-1}|$). So $q = e^{\pi i/n}$ for some integer $n \geq 2$, and we are done.

\[ \Box \]

\textbf{Corollary 1.45.16.} Let $A$ be a fusion ring, and $X \in A$ a basis element. Then if $FPdim(X) < 2$ then $FPdim(X) = 2\cos(\pi/n)$, for some integer $n \geq 3$.

\textit{Proof.} This follows from Proposition 1.45.15, since $FPdim(X^*) = FPdim(X)^2$. \[ \Box \]

\section{1.46. Deligne’s tensor product of finite abelian categories.} Let $\mathcal{C}, \mathcal{D}$ be two finite abelian categories over a field $k$.

\textbf{Definition 1.46.1.} Deligne’s tensor product $\mathcal{C} \boxtimes \mathcal{D}$ is an abelian category which is universal for the functor assigning to every $k$-linear abelian category $\mathcal{A}$ the category of right exact in both variables bilinear bifunctors $\mathcal{C} \times \mathcal{D} \to \mathcal{A}$. That is, there is a bifunctor $\boxtimes : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$ : $(X, Y) \mapsto X \boxtimes Y$ which is right exact in both variables and is such that for any right exact in both variables bifunctor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{A}$ there exists a unique right exact functor $\tilde{F} : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{A}$ satisfying $\tilde{F} \circ \boxtimes = F$.

\textbf{Proposition 1.46.2.} (cf. [D, Proposition 5.13]) (i) The tensor product $\mathcal{C} \boxtimes \mathcal{D}$ exists and is a finite abelian category.

(ii) It is unique up to a unique equivalence.

(iii) Let $\mathcal{C}, \mathcal{D}$ be finite dimensional algebras and let $\mathcal{C} = \mathcal{C} - \text{mod}$ and $\mathcal{D} = \mathcal{D} - \text{mod}$. Then $\mathcal{C} \boxtimes \mathcal{D} = \mathcal{C} \otimes \mathcal{D} - \text{mod}$.

(iv) The bifunctor $\boxtimes$ is exact in both variables and satisfies

$$\text{Hom}_\mathcal{C}(X_1, Y_1) \otimes \text{Hom}_\mathcal{D}(X_2, Y_2) \cong \text{Hom}_{\mathcal{C} \boxtimes \mathcal{D}}(X_1 \boxtimes X_2, Y_1 \boxtimes Y_2).$$

(v) any bilinear bifunctor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{A}$ exact in each variable defines an exact functor $\tilde{F} : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{A}$.

\textit{Proof.} (sketch). (ii) follows from the universal property in the usual way.

(i) As we know, a finite abelian category is equivalent to the category of finite dimensional modules over an algebra. So there exist finite dimensional algebras $\mathcal{C}, \mathcal{D}$ such that $\mathcal{C} = \mathcal{C} - \text{mod}$, $\mathcal{D} = \mathcal{D} - \text{mod}$. Then one can define $\mathcal{C} \boxtimes \mathcal{D} = \mathcal{C} \otimes \mathcal{D} - \text{mod}$, and it is easy to show that
it satisfies the required conditions. This together with (ii) also implies (iii).

(iv), (v) are routine. \(\square\)

A similar result is valid for locally finite categories.

Deligne’s tensor product can also be applied to functors. Namely, if \(F : \mathcal{C} \to \mathcal{C}'\) and \(G : \mathcal{D} \to \mathcal{D}'\) are additive right exact functors between finite abelian categories then one can define the functor \(F \boxtimes G : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{C}' \boxtimes \mathcal{D}'\).

**Proposition 1.46.3.** If \(\mathcal{C}, \mathcal{D}\) are multitensor categories then the category \(\mathcal{C} \boxtimes \mathcal{D}\) has a natural structure of a multitensor category.

**Proof.** Let \(X_1 \boxtimes Y_1, X_2 \boxtimes Y_2 \in \mathcal{C} \boxtimes \mathcal{D}\). Then we can set
\[
(X_1 \boxtimes Y_1) \otimes (X_2 \boxtimes Y_2) := (X_1 \otimes X_2) \boxtimes (Y_1 \boxtimes Y_2),
\]
and define the associativity isomorphism in the obvious way. This defines a structure of a monoidal category on the subcategory of \(\mathcal{C} \boxtimes \mathcal{D}\) consisting of “\(\boxtimes\)-decomposable” objects of the form \(X \boxtimes Y\). But any object of \(\mathcal{C} \boxtimes \mathcal{D}\) admits a resolution by \(\boxtimes\)-decomposable injective objects. This allows us to use a standard argument with resolutions to extend the tensor product to the entire category \(\mathcal{C} \boxtimes \mathcal{D}\). It is easy to see that if \(\mathcal{C}, \mathcal{D}\) are rigid, then so is \(\mathcal{C} \boxtimes \mathcal{D}\), which implies the statement. \(\square\)

1.47. **Finite (multi)tensor categories.** In this subsection we will study general properties of finite multitensor and tensor categories.

Recall that in a finite abelian category, every simple object \(X\) has a projective cover \(P(X)\). The object \(P(X)\) is unique up to a non-unique isomorphism. For any \(Y\) in \(\mathcal{C}\) one has

\[(1.47.1) \quad \dim \text{Hom}(P(X), Y) = [Y : X].\]

Let \(K_0(\mathcal{C})\) denote the free abelian group generated by isomorphism classes of indecomposable projective objects of a finite abelian category \(\mathcal{C}\). Elements of \(K_0(\mathcal{C}) \otimes \mathbb{Z} \mathbb{C}\) will be called virtual projective objects. We have an obvious homomorphism \(\gamma : K_0(\mathcal{C}) \to \text{Gr}(\mathcal{C})\). Although groups \(K_0(\mathcal{C})\) and \(\text{Gr}(\mathcal{C})\) have the same rank, in general \(\gamma\) is neither surjective nor injective even after tensoring with \(\mathbb{C}\). The matrix \(C\) of \(\gamma\) in the natural basis is called the Cartan matrix of \(\mathcal{C}\); its entries are \([P(X) : Y]\), where \(X, Y\) are simple objects of \(\mathcal{C}\).

Now let \(\mathcal{C}\) be a finite multitensor category, let \(I\) be the set of isomorphism classes of simple objects of \(\mathcal{C}\), and let \(i^*, \ast i\) denote the right and left duals to \(i\), respectively. Let \(\text{Gr}(\mathcal{C})\) be the Grothendieck ring of \(\mathcal{C}\), spanned by isomorphism classes of the simple objects \(X_i, i \in I\). In this ring, we have \(X_i X_j = \sum_k N_{ij}^k X_k\), where \(N_{ij}^k\) are nonnegative integers. Also, let \(P_i\) denote the projective covers of \(X_i\).
Proposition 1.47.1. Let $\mathcal{C}$ be a finite multitensor category. Then $K_0(\mathcal{C})$ is a $\text{Gr}(\mathcal{C})$-bimodule.

Proof. This follows from the fact that the tensor product of a projective object with any object is projective, Proposition 1.13.6. □

Let us describe this bimodule explicitly.

Proposition 1.47.2. For any object $Z$ of $\mathcal{C}$,

$$P_i \otimes Z \cong \oplus_{j,k} N_{kj}^i [Z : X_j] P_k, \quad Z \otimes P_i \cong \oplus_{j,k} N_{ik}^j [Z : X_j] P_k.$$

Proof. $\text{Hom}(P_i \otimes Z, X_k) = \text{Hom}(P_i, X_k \otimes Z^*)$, and the first formula follows from Proposition 1.13.6. The second formula is analogous. □

Proposition 1.47.3. Let $P$ be a projective object in a multitensor category $\mathcal{C}$. Then $P^*$ is also projective. Hence, any projective object in a multitensor category is also injective.

Proof. We need to show that the functor $\text{Hom}(P^*, \bullet)$ is exact. This functor is isomorphic to $\text{Hom}(1, P \otimes \bullet)$. The functor $P \otimes \bullet$ is exact and moreover, by Proposition 1.13.6, any exact sequence splits after tensoring with $P$, as an exact sequence consisting of projective objects. The Proposition is proved. □

Proposition 1.47.3 implies that an indecomposable projective object $P$ has a unique simple subobject, i.e., that the socle of $P$ is simple.

For any finite tensor category $\mathcal{C}$ define an element $R_\mathcal{C} \in K_0(\mathcal{C}) \otimes \mathbb{C}$ by

$$(1.47.2) \quad R_\mathcal{C} = \sum_{i \in I} \text{FPdim}(X_i) P_i.$$

Definition 1.47.4. The virtual projective object $R_\mathcal{C}$ is called the regular object of $\mathcal{C}$.

Definition 1.47.5. Let $\mathcal{C}$ be a finite tensor category. Then the Frobenius-Perron dimension of $\mathcal{C}$ is defined by

$$(1.47.3) \quad \text{FPdim}(\mathcal{C}) := \text{FPdim}(R_\mathcal{C}) = \sum_{i \in I} \text{FPdim}(X_i) \text{FPdim}(P_i).$$

Example 1.47.6. Let $H$ be a finite dimensional quasi-Hopf algebra. Then $\text{FPdim}(\text{Rep}(H)) = \dim(H)$.

Proposition 1.47.7. (1) $Z \otimes R_\mathcal{C} = R_\mathcal{C} \otimes Z = \text{FPdim}(Z) R_\mathcal{C}$ for all $Z \in \text{Gr}(\mathcal{C})$.

(2) The image of $R_\mathcal{C}$ in $\text{Gr}(\mathcal{C}) \otimes \mathbb{C}$ is a regular element.
Proof. We have \( \sum_i \text{FPdim}(X_i) \dim \text{Hom}(P_i, Z) = \text{FPdim}(Z) \) for any object \( Z \) of \( \mathcal{C} \). Hence,
\[
\sum_i \text{FPdim}(X_i) \dim \text{Hom}(P_i \otimes Z, Y) = \sum_i \text{FPdim}(X_i) \dim \text{Hom}(P_i, Y \otimes Z^*)
\]
\[
= \text{FPdim}(Y \otimes Z^*)
\]
\[
= \text{FPdim}(Y) \text{FPdim}(Z^*)
\]
\[
= \text{FPdim}(Y) \text{FPdim}(Z)
\]
\[
= \text{FPdim}(Z) \sum_i \text{FPdim}(X_i) \dim \text{Hom}(P_i, Y).
\]

Now, \( P(X) \otimes Z \) are projective objects by Proposition 1.13.6. Hence, the formal sums \( \sum_i \text{FPdim}(X_i) P_i \otimes Z = R \otimes Z \) and \( \text{FPdim}(Z) \sum_i \text{FPdim}(X_i) P_i = \text{FPdim}(Z) R \) are linear combinations of \( P_j, j \in I \) with the same coefficients. \( \square \)

Remark 1.47.8. We note the following useful inequality:
\[
(1.47.4) \quad \text{FPdim}(C) \geq N \text{FPdim}(P),
\]
where \( N \) is the number of simple objects in \( C \), and \( P \) is the projective cover of the neutral object \( 1 \). Indeed, for any simple object \( V \) the projective object \( P(V) \otimes ^*V \) has a nontrivial homomorphism to \( 1 \), and hence contains \( P \). So \( \text{FPdim}(P(V)) \text{FPdim}(V) \geq \text{FPdim}(P) \). Adding these inequalities over all simple \( V \), we get the result.

1.48. Integral tensor categories.

Definition 1.48.1. A transitive unital \( \mathbb{Z}_+ \)-ring \( A \) of finite rank is said to be integral if \( \text{FPdim} : A \to \mathbb{Z} \) (i.e. the Frobenius-Perron dimensions of elements of \( C \) are integers). A tensor category \( C \) is integral if \( \text{Gr}(C) \) is integral.

Proposition 1.48.2. A finite tensor category \( C \) is integral if and only if \( C \) is equivalent to the representation category of a finite dimensional quasi-Hopf algebra.

Proof. The “if” part is clear from Example 1.45.6. To prove the “only if” part, it is enough to construct a quasi-fiber functor on \( C \). Define \( P = \bigoplus_i \text{FPdim}(X_i) P_i \), where \( X_i \) are the simple objects of \( C \), and \( P_i \) are their projective covers. Define \( F = \text{Hom}(P, \bullet) \). Obviously, \( F \) is exact and faithful, \( F(1) \cong 1 \), and \( \dim F(X) = \text{FPdim}(X) \) for all \( X \in C \). Using Proposition 1.46.2, we continue the functors \( F(\bullet \otimes \bullet) \) and \( F(\bullet) \otimes F(\bullet) \) to the functors \( C \otimes C \to \text{Vec} \). Both of these functors are exact and take the same values on the simple objects of \( C \otimes C \). Thus these functors are isomorphic and we are done. \( \square \)
Corollary 1.48.3. The assignment $H \mapsto \text{Rep}(H)$ defines a bijection between integral finite tensor categories $\mathcal{C}$ over $k$ up to monoidal equivalence, and finite dimensional quasi-Hopf algebras $H$ over $k$, up to twist equivalence and isomorphism.

1.49. **Surjective quasi-tensor functors.** Let $\mathcal{C}, \mathcal{D}$ be abelian categories. Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor.

**Definition 1.49.1.** We will say that $F$ is **surjective** if any object of $\mathcal{D}$ is a subquotient in $F(X)$ for some $X \in \mathcal{C}$. \(^\text{13}\)

**Exercise 1.49.2.** Let $A, B$ be coalgebras, and $f : A \to B$ a homomorphism. Let $F = f^* : A - \text{comod} \to B - \text{comod}$ be the corresponding pushforward functor. Then $F$ is surjective if and only if $f$ is surjective.

Now let $\mathcal{C}, \mathcal{D}$ be finite tensor categories.

**Theorem 1.49.3.** ([EO]) Let $F : \mathcal{C} \to \mathcal{D}$ be a surjective quasi-tensor functor. Then $F$ maps projective objects to projective ones.

**Proof.** Let $\mathcal{C}$ be a finite tensor category, and $X \in \mathcal{C}$. Let us write $X$ as a direct sum of indecomposable objects (such a representation is unique). Define the **projectivity defect** $p(X)$ of $X$ to be the sum of Frobenius-Perron dimensions of all the non-projective summands in this sum (this is well defined by the Krull-Schmidt theorem). It is clear that $p(X \oplus Y) = p(X) + p(Y)$. Also, it follows from Proposition 1.13.6 that $p(X \otimes Y) \leq p(X)p(Y)$.

Let $P_i$ be the indecomposable projective objects in $\mathcal{C}$. Let $P_i \otimes P_j \cong \oplus_k B_{ij}^k P_k$, and let $B_i$ be the matrix with entries $B_{ij}^k$. Also, let $B = \sum B_i$. Obviously, $B$ has strictly positive entries, and the Frobenius-Perron eigenvalue of $B$ is $\sum_i \text{FPdim}(P_i)$.

On the other hand, let $F : \mathcal{C} \to \mathcal{D}$ be a surjective quasi-tensor functor between finite tensor categories. Let $p_j = p(F(P_j))$, and $p$ be the vector with entries $p_j$. Then we get $p_ip_j \geq \sum_k B_{ij}^k p_k$, so $(\sum_i p_i)p \geq Bp$. So, either $p_i$ are all zero, or they are all positive, and the norm of $B$ with respect to the norm $|x| = \sum p_i |x_i|$ is at most $\sum p_i$. Since $p_i \leq \text{FPdim}(P_i)$, this implies $p_i = \text{FPdim}(P_i)$ for all $i$ (as the largest eigenvalue of $B$ is $\sum_i \text{FPdim}(P_i)$).

Assume the second option is the case. Then $F(P_i)$ do not contain nonzero projective objects as direct summands, and hence for any projective $P \in \mathcal{C}$, $F(P)$ cannot contain a nonzero projective object as a direct summand. However, let $Q$ be a projective object of $\mathcal{D}$. Then,

\(^\text{13}\) This definition does not coincide with a usual categorical definition of surjectivity of functors which requires that every object of $\mathcal{D}$ be isomorphic to some $F(X)$ for an object $X$ in $\mathcal{C}$. 
since $F$ is surjective, there exists an object $X \in \mathcal{C}$ such that $Q$ is a subquotient of $F(X)$. Since any $X$ is a quotient of a projective object, and $F$ is exact, we may assume that $X = P$ is projective. So $Q$ occurs as a subquotient in $F(P)$. As $Q$ is both projective and injective, it is actually a direct summand in $F(P)$. Contradiction.

Thus, $p_i = 0$ and $F(P_i)$ are projective. The theorem is proved. \hfill \Box

1.50. **Categorical freeness.** Let $\mathcal{C}, \mathcal{D}$ be finite tensor categories, and $F : \mathcal{C} \to \mathcal{D}$ be a quasi-tensor functor.

**Theorem 1.50.1.** One has

\[(1.50.1) \quad F(R_\mathcal{C}) = \frac{\text{FPdim(}\mathcal{C})}{\text{FPdim(}\mathcal{D})} R_\mathcal{D}.\]

**Proof.** By Theorem 1.49.3, $F(R_\mathcal{C})$ is a virtually projective object. Thus, $F(R_\mathcal{C})$ must be proportional to $R_\mathcal{D}$, since both (when written in the basis $P_i$) are eigenvectors of a matrix with strictly positive entries with its Frobenius-Perron eigenvalue. (For this matrix we may take the matrix of multiplication by $F(X)$, where $X$ is such that $F(X)$ contains as composition factors all simple objects of $\mathcal{D}$; such exists by the surjectivity of $F$). The coefficient is obtained by computing the Frobenius-Perron dimensions of both sides. \hfill \Box

**Corollary 1.50.2.** In the above situation, one has $\text{FPdim(}\mathcal{C}) \geq \text{FPdim(}\mathcal{D})$, and $\text{FPdim(}\mathcal{D})$ divides $\text{FPdim(}\mathcal{C})$ in the ring of algebraic integers. In fact,

\[(1.50.2) \quad \frac{\text{FPdim(}\mathcal{C})}{\text{FPdim(}\mathcal{D})} = \sum \text{FPdim}(X_i) \dim \text{Hom}(F(P_i), 1_\mathcal{D}),\]

where $X_i$ runs over simple objects of $\mathcal{C}$.

**Proof.** The statement is obtained by computing the dimension of $\text{Hom}(\bullet, 1_\mathcal{D})$ for both sides of (1.50.1). \hfill \Box

Suppose now that $\mathcal{C}$ is integral, i.e., by Proposition 1.48.2, it is the representation category of a quasi-Hopf algebra $H$. In this case, $R_\mathcal{C}$ is an honest (not only virtual) projective object of $\mathcal{C}$, namely the free rank 1 module over $H$. Therefore, multiples of $R_\mathcal{C}$ are free $H$-modules of finite rank, and vice versa.

Then Theorem 1.49.3 and the fact that $F(R_\mathcal{C})$ is proportional to $R_\mathcal{D}$ implies the following categorical freeness result.

**Corollary 1.50.3.** If $\mathcal{C}$ is integral, and $F : \mathcal{C} \to \mathcal{D}$ is a surjective quasi-tensor functor then $\mathcal{D}$ is also integral, and the object $F(R_\mathcal{C})$ is free of rank $\text{FPdim(}\mathcal{C})/\text{FPdim(}\mathcal{D})$ (which is an integer).
Proof. The Frobenius-Perron dimensions of simple objects of $\mathcal{D}$ are coordinates of the unique eigenvector of the positive integer matrix of multiplication by $F(R_C)$ with integer eigenvalue $FPdim(C)$, normalized so that the component of $1$ is $1$. Thus, all coordinates of this vector are rational numbers, hence integers (because they are algebraic integers). This implies that the category $\mathcal{D}$ is integral. The second statement is clear from the above. \qed

Corollary 1.50.4. ([Scha]; for the semisimple case see [ENO1]) A finite dimensional quasi-Hopf algebra is a free module over its quasi-Hopf subalgebra.

Remark 1.50.5. In the Hopf case Corollary 1.50.3 is well known and much used; it is due to Nichols and Zoeller [NZ].