1. It’s enough to solve the congruence mod 11 and mod 13, and then combine the solutions by Chinese Remainder Theorem. Now $x^3 - 9x^2 + 23x - 15$ factors as $(x - 1)(x - 3)(x - 5)$, so solutions mod 11 or mod 13 are 1, 3, 5 in each case. To combine, we first need $x, y$ such that $13x + 11y = 1$. For instance $x = -5, y = 6$ works. (We can find $x, y$ by Euclidean algorithm). So if we have a solution $a$ mod 11 and a solution $b$ mod 13 then the Chinese Remainder Theorem recipe tells us that

$$(5)(13)a + (6)(11)b = -65a + 66b$$

is a solution mod 143. Running this over $a \in \{1, 3, 5\}$ and $b \in \{1, 3, 5\}$ we get 9 solutions: 1, 3, 5, 14, 16, 27, 122, 133, 135.

2. We just need to compute these expressions mod 4 and mod 25, and then combine using CRT. Note that (1)(25) + (−6)(4) = 1, so if $x \equiv a \pmod{4}$ and $x \equiv b \pmod{25}$ then $x \equiv 25a - 24b \pmod{100}$.

For $2^{100}$: We have $2^{100} \equiv 0 \pmod{4}$ and $2^{100} = 2^{\varphi(25)} \equiv 1 \pmod{25}$. So the last two digits are $25 \cdot 0 - 24 \cdot 1 \equiv 76$.

For $3^{100}$: We have $3^{100} = 3^{50 \varphi(4)} \equiv 1 \pmod{4}$ and $3^{100} = 3^{50 \varphi(25)} \equiv 1 \pmod{25}$. So the last two digits are $25 \cdot 1 - 24 \cdot 1 \equiv 01$.

3. Let $m = \prod p_i^{e_i}$. By the CRT, we can simply find the number of solutions mod $p_i^{e_i}$ for each $i$ and take the product. Now $x^2 \equiv x \pmod{p^r}$ means $p^r | x^2 - x = x(x - 1)$. Since $x$ and $x - 1$ are coprime, we have $p^r | x$ or $p^r | x - 1$. So $x \equiv 0, 1 \pmod{p^r}$ are the two solutions. Thus, for an arbitrary integer $m$, the number of solutions is $2^r$ where $r$ is the number of distinct prime divisors of $m$.

4. (a) We need to show that $a^{560} \equiv 1 \pmod{3}$, mod 11, and mod 17 for any $a$ coprime to 561.

Since $a$ is coprime to 3, $a^2 \equiv 1 \pmod{3}$, so $a^{560} = a^{2^{280}} \equiv 1 \pmod{3}$.

Since $a$ is coprime to 11, $a^{10} \equiv 1 \pmod{11}$, so $a^{560} = a^{56 \cdot 10} \equiv 1 \pmod{11}$.

Since $a$ is coprime to 17, $a^{16} \equiv 1 \pmod{17}$, so $a^{560} = a^{35 \cdot 16} \equiv 1 \pmod{17}$.

(b) Suppose $n = pq$ with $p, q$ distinct primes satisfies property $P$. Then for all $a$ coprime to $p$ and $q$, we have $a^{pq-1} \equiv 1 \pmod{p}$ and $a^{pq-1} \equiv 1 \pmod{q}$.

Assume, without loss of generality, that $p < q$. Then

$$a^{pq-1} = a^{(q-1)p + p-1} = a^{(q-1)p} \cdot a^{p-1} \equiv 1^p \cdot a^{p-1} \pmod{q}.$$  

Now for any $x$ coprime to $q$, we can let $a$ be the unique integer mod $pq$ which satisfies $a \equiv x \pmod{q}$ and $a \equiv 1 \pmod{p}$, so that $a$ is coprime to $pq$ and thus $x^{p-1} \equiv 1 \pmod{q}$. However, because of the existence of a primitive root mod $q$, we know that $q - 1$ is the smallest positive integer such that $x^{q-1} \equiv 1 \pmod{q}$ for every $x$ coprime to $q$. Since $p - 1 < q - 1$, we have a contradiction.

(c) A sufficient condition is that $p-1 | pqr-1$. This implies that $qr \equiv 1 \pmod{p-1}, pr \equiv 1 \pmod{q-1},$
and \( pq \equiv 1 \pmod{r-1} \). Using it to search we find the following numbers:

\[
\begin{align*}
561 &= 3 \cdot 11 \cdot 17 \\
1105 &= 5 \cdot 13 \cdot 17 \\
1729 &= 7 \cdot 13 \cdot 19 \\
2465 &= 5 \cdot 17 \cdot 29 \\
2821 &= 7 \cdot 13 \cdot 31 \\
6601 &= 7 \cdot 23 \cdot 41 \\
8911 &= 7 \cdot 19 \cdot 67 \\
10585 &= 5 \cdot 29 \cdot 73 \\
15841 &= 7 \cdot 31 \cdot 73 \\
29341 &= 13 \cdot 37 \cdot 61.
\end{align*}
\]

5. Yes. Pick distinct primes \( p_1, \ldots, p_N \) and let \( x \) solve

\[
\begin{align*}
x &\equiv 0 \pmod{p_1^2} \\
x + 1 &\equiv 0 \pmod{p_2^2} \\
& \vdots \\
x + N - 1 &\equiv 0 \pmod{p_N^2}
\end{align*}
\]

This has solutions mod \( p_1^2 \cdots p_N^2 \), by CRT. We can pick \( x \) positive. Then for each \( i \), \( x + i - 1 \) is divisible by \( p_i^2 \), and thus is not squarefree.

6. (a) You should find that the density is about 2/3.
(b) You should find that the density is about 1/3.
(c) The key difference is the Galois group, which is \( S_3 \) for (a) and \( \mathbb{Z}/3\mathbb{Z} \) for (b). The reason for the distribution you see is a deep theorem in algebraic number theory called the Chebotarev density theorem. In terms of group theory, the main difference is that the number of permutations in \( S_3 \) with a fixed point is 4, leading to the fraction \( 4/6 = 2/3 \), while the corresponding number for \( A_3 = \{(1), (123), (132)\} \) is 1, leading to the fraction 1/3.