Lecture 19
Continued Fractions II: Inequalities

Real number $x$, compute integers $a_0, a_1, \ldots$ such that $a_0 = \lfloor x \rfloor$,

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}.$$

Let $x_1 = \frac{1}{x-a_0}$, real number $\geq 1$ as long as well defined, $a_1 = \lfloor x_1 \rfloor$, $x_2 = \frac{1}{x_1-a_1}$.

For $i > 0$, $a_i \geq 1$.

Convergents $\frac{p_n}{q_n} = [a_0, a_1, \ldots a_n]$. $\frac{p_n}{q_n} \to x$ as $n \to \infty$.

$$\frac{p_0}{q_0} < \frac{p_2}{q_4} < \ldots < x < \ldots < \frac{p_3}{q_3} < \frac{p_1}{q_1}$$

$$\left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| = \frac{1}{q_n q_{n+1}} \to 0 \text{ as } n \to \infty$$

so

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} \leq \frac{1}{q_n^2}$$

Why are continued fractions useful/interesting?

1. Gives good approximations to real numbers
2. Continued fractions and higher dimensional variants have applications in engineering
3. Useful in number theory for study of quadratic fields, diophantine equations

**Theorem 66.** One of every two consecutive convergents satisfies

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{2q_n^2}$$

**Proof.**

$$\left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| = \frac{1}{q_n q_{n+1}} \leq \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2}$$

using AM-GM inequality with $\frac{1}{q_n}$ and $\frac{1}{q_{n+1}}$

$$\left| x - \frac{p_n}{q_n} \right| + \left| x - \frac{p_{n+1}}{q_{n+1}} \right| = \left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| \leq \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2}$$
\[ \Rightarrow \left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{2q_n^2} \text{ or } \left| x - \frac{p_{n+1}}{q_{n+1}} \right| \leq \frac{1}{2q_{n+1}^2} \]

**Theorem 67.** One of every three consecutive convergents satisfies

\[ \left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{\sqrt{5}q_n^2} \]

**Proof.** Suppose not, and that

\[ \left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{\sqrt{5}q_n^2} \text{ for } n, n + 1, n + 2 \]

\[ \left| x - \frac{p_n}{q_n} \right| + \left| x - \frac{p_{n+1}}{q_{n+1}} \right| = \left| \frac{p_n - p_{n+1}}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| \]

\[ = \frac{1}{q_nq_{n+1}} \]

\[ > \frac{1}{\sqrt{5}q_n^2} + \frac{1}{\sqrt{5}q_{n+1}^2} \]

\[ \Rightarrow \sqrt{5} > \frac{q_{n+1}}{q_n} + \frac{q_n}{q_{n+1}} \]

\[ \Rightarrow \frac{q_{n+1}}{q_n} < \sqrt{5} + \frac{1}{2} \]

using the fact that \( f(x) = x + \frac{1}{x} \) is strictly increasing on \((1, \infty)\)

\[ \Rightarrow \frac{q_n}{q_{n+1}} = \frac{\frac{q_n}{q_{n+1}}}{\frac{q_n}{q_{n+1}} + \frac{q_n}{q_{n+1}}} = \frac{\sqrt{5} - 1}{2} \]

Same argument for \( n + 1 \) and \( n + 2 \) says that \( \frac{q_{n+2}}{q_{n+1}} < \frac{\sqrt{5} + 1}{2} \).

\[ \frac{q_{n+2}}{q_{n+1}} = \frac{a_{n+2}q_n + q_n}{q_{n+1}} \]

\[ = a_{n+2} + \frac{q_n}{q_{n+1}} \]

\[ \geq 1 + \frac{\sqrt{5} - 1}{2} \]

\[ = \frac{\sqrt{5} + 1}{2} \]

leading to a contradiction (\( \ddagger \))
**Corollary 68.** For any irrational real number \( x \), there are infinitely many rational numbers \( \frac{p}{q} \) such that \( |x - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2} \)

**Proof.** Write convergents as

\[
\begin{align*}
p_1 & \ p_2 & \ p_3 & \ p_4 & \ p_5 & \ p_6 & \ldots \\
q_1 & \ q_2 & \ q_3 & \ q_4 & \ q_5 & \ q_6 & \ldots \\
& \text{one satisfies} & \text{one satisfies}
\end{align*}
\]

\[\square\]

**Theorem 69.** \( \sqrt{5} \) is optimal (cannot be replaced by any larger value) - ie., there does not exist an \( \alpha < \sqrt{5} \) such that for any irrational \( x \) there are infinitely many rational numbers \( |x - \frac{p}{q}| < \frac{1}{\alpha q^2} \)

**Proof.** We won’t prove this here [proved in PSet 9], but we’ll give a heuristic argument for why \( \sqrt{5} \) is the best.

Consider \( \alpha = \frac{1 + \sqrt{5}}{2} = \text{golden ratio} \). It has the continued fraction \([1, 1, 1, \ldots]\), and convergents are \( 1, 1 + \frac{1}{1} = 2, 1 + \frac{1}{2} = \frac{3}{2}, 1 + \frac{1}{3} = \frac{5}{3}, 1 + \frac{1}{5} = \frac{8}{5} \ldots \) By induction they are ratios of consecutive Fibonacci numbers.

\[F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}\]

We’ll show

\[
\left| \frac{F_{n+1}}{F_n} - \alpha \cdot F_n^2 \right| \to \frac{1}{\alpha - \beta} = \frac{1}{\sqrt{5}} \quad \text{as } n \to \infty
\]

\[
\left| \frac{F_{n+1}}{F_n} - \alpha \cdot F_n^2 \right| = \left| \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} - \alpha \right| \cdot \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right)^2
\]

\[= \left| \frac{\alpha^{n+1} - \beta^{n+1} - \alpha^{n+1} + \beta^n \alpha}{\alpha^n - \beta^n} \right| \cdot \frac{|\alpha^n - \beta^n|^2}{|\alpha - \beta|^2}
\]

\[= \frac{\beta^n |\alpha - \beta|}{|\alpha - \beta|^2}
\]

\[= \frac{|(\beta \alpha)^n - \beta^{2n}|}{|\alpha - \beta|}
\]

\[= \frac{|(-1)^n - \beta^{2n}|}{|\alpha - \beta|}
\]

Since \( |\beta| < 1, \beta^{2n} \to 0 \) as \( n \to \infty \), so expression tends to \( \frac{1}{\alpha - \beta} = \frac{1}{\sqrt{5}} \) as \( n \to \infty \). \[\square\]
Theorem 70. A real number $x$ is a quadratic irrational (i.e., $x = r + s\sqrt{t}$ where $r, s \in \mathbb{Q}$, and $t$ is a squarefree integer) if and only if its continued fraction is periodic ($x = [b_0, b_1, \ldots, b_k, a_0, a_1, \ldots a_{n-1}, a_0, a_1, \ldots a_{n-1}, \ldots] = [b_0, \ldots, b_k, a_0, \ldots, a_{n-1}]$.

Proof - Part 1. Suppose $x = [b_0, b_1, \ldots, b_k, a_0, a_1, \ldots a_{n-1}], let \theta = [a_0, a_1, \ldots a_{n-1}].$

\[ \theta = [a_0, a_1, \ldots a_{n-1}, \theta] = \frac{p_{n-1} \theta + p_{n-2}}{q_{n-1} \theta + q_{n-2}} \]

for some positive $p_{n-1}, p_{n-2}, q_{n-1}, q_{n-2}$, which leads to a quadratic equation for $\theta$. $\theta$ irrational because it’s an infinite continued fraction. Then $x = [b_0, \ldots, b_k, \theta]$ is also a quadratic irrational.

Proof - Part 2. Want to show that if $x = \frac{a + \sqrt{b}}{c}$, where $a, b, c$ are integers, $b > 0$, $c \neq 0$, $b$ not a perfect square, then continued fraction of $x$ is periodic.

Step 0: We can write this as

\[ x = \frac{ac + \sqrt{bc^2}}{c^2} \quad \text{if } c > 0, \quad \text{or} \quad \frac{-ac + \sqrt{bc^2}}{-c^2} \quad \text{if } c < 0 \]

In either case, $bc^2 - (\pm ac)^2 = c^2(b - a^2)$, which is divisible by $\pm c^2$. In either case, we’ve written $x$ as

\[ x = \frac{B_0 + \sqrt{d}}{C_0}, \quad \text{with } C|d - B_0^2 \]

Fix such an expression (in particular, $d$).

Step 1: Let $x_0 = x$ and define by induction

\[ a_i = \lfloor x_i \rfloor \]
\[ x_i = \frac{B_i + \sqrt{d}}{C_i} \]
\[ x_{i+1} = \frac{1}{x_i - a_i} \]
\[ B_{i+1} = a_i C_i - B_i \]
\[ C_{i+1} = \frac{d - B_i^2}{C_i} \]

So far, we know $B_i, C_i$ are rational numbers. Strategy will be to show that $B_i, C_i$ are integers, and that they’re bounded in absolute value - use this to show that they repeat.
Definitions of $B_{i+1}, C_{i+1}$ motivated by

\[
x_i = \frac{B_i + \sqrt{d}}{C_i}
\]

\[
x_{i+1} = \frac{1}{x_i - a_i}
= \frac{1}{\frac{B_i + \sqrt{d}}{C_i} - a_i}
= \frac{C_i}{\sqrt{d} - (a_iC_i - B_i)}
\]

\[
\frac{B_{i+1} + \sqrt{d}}{C_{i+1}} = \frac{C_i(\sqrt{d} + a_iC_i - B_i)}{d - (a_iC_i - B_i)^2}
= \frac{a_iC_i - B_i + \sqrt{d}}{\frac{d - B_{i+1}^2}{C_i}}
\]