These problems are related to the material covered in Lectures 21-22. I have made every effort to proof-read them, but some errors may remain. The first person to spot each error will receive 1-5 points of extra credit.

The problem set is due by the start of class on 12/3/2013 and should be submitted electronically as a pdf-file e-mailed to the instructor. You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions and to identify collaborators and any sources not listed in the syllabus.

Recall that we have defined a curve as a smooth projective variety of dimension one (and varieties are defined to be irreducible algebraic sets).

**Problem 1. Bezout’s theorem (50 points)**

In this problem $k$ is an algebraically closed field.

A curve in $\mathbb{P}^2$ is called a plane curve.¹

(a) Prove that every plane curve $X/k$ is a hypersurface, meaning that its ideal $I(X)$ is of the form $(f)$, where $f$ is a homogeneous polynomial in $k[x,y,z]$. Then show that every generator for $I(X)$ has the same degree.

The degree of $X$ (denoted $\deg X$) is the degree of any generator for its homogeneous ideal.

(b) Let $F/k$ be a function field, let $P$ be a place of $F$, and let $f \in \mathcal{O}_P$. Prove that the ring $\mathcal{O}_P/(f)$ is a $k$-vector space of dimension $\ord_P(f)$.

Given a nonconstant homogeneous polynomial $g \in k[x,y,z]$ that is relatively prime to $f$, we can represent $g$ as an element of the local ring $\mathcal{O}_{X,P}$ of functions in $X$ that are regular at $P$ by picking a homogeneous polynomial $h$ that does not vanish at $P$ and representing $g$ as $g/h$ reduced modulo $I(X)$, an element of $k(X)$. Note that in terms of computing $\ord_P(g)$ it makes no difference which $h$ we pick, $\ord_P(g)$ will always be equal to the order of vanishing of $g$ at $P$, a nonnegative integer. We then define the divisor of $g$ in $\text{Div}_k X$ to be

$$\text{div}_X g = \sum \ord_P(g) P.$$ 

Note that $\text{div}_X g$ is not a principal divisor.² Indeed, $\deg \text{div}_X g$ is never zero.

(c) Prove that $\deg \text{div}_X g$ depends only on $\deg g$ (i.e. $\deg \text{div}_X g = \deg \text{div}_X h$ whenever $g$ and $h$ have the same degree and are both relatively prime to $f$). Then prove that $\deg \text{div}_X g$ is a linear function of $\deg g$.

Now suppose that $g$ is irreducible and nonsingular, so it defines a plane curve $Y/k$.

(d) Prove that $\deg \text{div}_Y f = \deg \text{div}_X g$.

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¹Plane curves are not usually required to be smooth or irreducible, but ours are.
²By varying $h$ locally we eliminate the poles that would be present if we fixed a global choice for $h$. 
**Definition 1.** Let \( f \) and \( g \) be two nonconstant homogeneous polynomials in \( k[x, y, z] \) with no common factor, and let \( P \) be a point in \( \mathbb{P}^2 \). The **intersection number** of \( f \) and \( g \) at \( P \) is

\[
I_P(f, g) := \text{dim}_k \mathcal{O}_{\mathbb{P}^2, P}/(f, g)
\]

Here \( \mathcal{O}_{\mathbb{P}^2, P} \) denotes the ring of functions in \( k(\mathbb{P}^2) \) that are regular at \( P \), and \( f \) and \( g \) are represented as elements of this ring by choosing homogeneous denominators of appropriate degree that do not vanish at \( P \), exactly as described above.

As above, let \( X/k \) and \( Y/k \) denote plane curves defined by relatively prime homogeneous polynomials \( f \) and \( g \), and let \( I(f, g) = \sum P I_P(f, g) \).

(e) Prove that \( I(f, g) \) is equal to \( \text{deg div}_X g = \text{deg div}_Y f \).

(f) Prove **Bezout’s Theorem** for plane curves:

\[
I(f, g) = \text{deg } f \text{ deg } g.
\]

In fact Bezout’s theorem holds even when \( f \) and \( g \) are not necessarily irreducible and nonsingular, but you need not prove this. It should be clear that \( f \) and \( g \) do not need to be irreducible; just factor them and apply the theorem to all pairs of factors. You proof should also handle cases where just one of \( f \) or \( g \) is singular; it takes a bit more work to handle the case where both \( f \) and \( g \) are singular and intersect at a common singularity. The assumption that \( k = \overline{k} \) is necessary, in general, but the inequality \( I(f, g) \leq \text{deg } f \text{ deg } g \) always holds.

**Problem 2. Derivations and differentials (50 points)**

A **derivation** on a function field \( F/k \) is a \( k \)-linear map \( \delta : F \to F \) such that

\[
\delta(fg) = \delta(f)g + f\delta(g).
\]

for all \( f, g \in F \).

(a) Prove that the following hold for any derivation \( \delta \) on \( F/k \):

(i) \( \delta(c) = 0 \) for all \( c \in k \).

(ii) \( \delta(f^n) = nf^{n-1}\delta(f) \) for all \( f \in F^\times \) and \( n \in \mathbb{Z} \).

(iii) If \( k \) has positive characteristic \( p \) then \( \delta(f^p) = 0 \) for all \( f \in F \).

(iv) \( \delta(f/g) = (\delta(f)g - f\delta(g))/g^2 \) for all \( f, g \in F \) with \( g \neq 0 \).

To simplify matters, we henceforth assume that \( k \) has characteristic zero.\(^3\)

The simplest example of a derivation is in the case where \( F = k(x) \) is the rational function field and \( \delta : F \to F \) is the map defined by \( \delta(f) = \partial f/\partial x \). We want to generalize this example to arbitrary function fields.

\(^3\)For those who are interested, the key thing that changes in characteristic \( p > 0 \) is that everywhere we require an element \( x \) to be transcendental we need to additionally require it to be a **separating element**, which means that \( F/k(x) \) is a separable extension.
Let $x$ be a transcendental element of $F/k$. Any $y \in F$ is then algebraic over $k(x)$ and has a minimal polynomial $\lambda \in k(x)[T]$. After clearing denominators we can assume that $\lambda \in k[x,T]$. We now formally define

$$\frac{\partial y}{\partial x} := -\frac{\partial \lambda/\partial x}{\partial \lambda/\partial T}(y) \in k(x,y) \subseteq F$$

and let the map $\delta_x : F \to F$ send $y$ to $\partial y/\partial x$.

One can show (but you are not asked to do this) that $\delta_x$ is a derivation on $F/k$. Note that we get a derivation $\delta_x$ for each transcendental $x$ in $F$. Now let $D_F$ be the set of all derivations on $F/k$.

(b) Let $x$ be a transcendental element of $F/k$. Prove that for any $\delta_1, \delta_2 \in D_F$ we have $\delta_1(x) = \delta_2(x) \Rightarrow \delta_1 = \delta_2$. Conclude that $\delta_x$ is the unique $\delta \in D_F$ for which $\delta(x) = 1$.

(c) Prove the following:

(i) For all $\delta_1, \delta_2 \in D_F$ the map $(\delta_1 + \delta_2) : F \to F$ defined by $f \mapsto \delta_1(f) + \delta_2(f)$ is a derivation (hence an element of $D_F$).

(ii) For all $f \in F$ and $\delta \in D_F$ the map $(f\delta) : F \to F$ defined by $g \mapsto f\delta(g)$ is a derivation (hence an element of $D_F$).

(iii) Every $\delta \in D_F$ satisfies $\delta = \delta(x)\delta_x$ (in particular, the chain rule $\delta_y = \delta_y(x)\delta_x$ holds for any transcendental $x, y \in F/k$).

It follows that we may view $D_F$ as one-dimensional $F$-vector space with any $\delta_x$ as a basis vector. But rather than fixing a particular basis vector; instead, let us define a relation on the set $S$ of pairs $(u, x)$ with $u, x \in F$ and $x$ transcendental over $k$:

$$(u, x) ~ (v, y) \iff v = u\delta_y(x). \quad (1)$$

(d) Prove that $\sim$ is an equivalence relation on $S$.

For each transcendental element $x \in F/k$, let the symbol $dx$ denote the equivalence class of $(1, x)$, and for $u \in F$ define $udx$ to be the equivalence class of $(u, x)$; we call $dx$ a differential. It follows from part (iii) of (d) that every derivation $\delta$ can be uniquely represented as $\delta = udx$ for some $u \in F$, but now we have the freedom to change representations; we may also write $\delta = vdy$ for any transcendental element $y$, where $v = u\delta_y(x) = u\partial x/\partial y$.

(e) Prove that $d(x+y) = dx+dy$ and $d(xy) = xdy+ydx$ for all transcendental $x, y \in F/k$.

Let us now extend our differential notation to elements of $F$ that are not transcendental over $k$. Recall that $k$ is algebraically closed in $F$, so we only need to consider elements of $k$.

(f) Prove that defining $da = 0$ for all $a \in k$ ensures that (e) holds for all $x, y \in F$, and that no other choice does.

Now momentarily forget everything above and just define $\Delta_F$ to the $F$-vector space generated by the set of formal symbols $\{dx : x \in F\}$, subject to the relations

$$\begin{align*}
(1) & \quad d(x+y) = dx + dy, \quad (2) d(xy) = xdy + ydx, \quad (3) da = 0 \text{ for } a \in k.
\end{align*}$$

Note that $x$ and $y$ denote elements of $F$ (functions), not free variables, so $\Delta_F$ reflects the structure of $F$ and will be different for different function fields.
(g) Prove that dim$_F$ $\Delta_F = 1$, and that any $dx$ with $x \notin k$ is a basis.

The set $\Delta = \Delta_F$ is often used as an alternative to the set of Weil differentials $\Omega$. They are both one-dimensional $F$-vector spaces, hence isomorphic (as $F$-vector spaces). But in order to be useful, we need to associate divisors to differentials in $\Delta$, as we did for $\Omega$.

For any differential $\omega \in \Delta$ and any place $P$, we may pick a uniformizer $t$ for $P$ and write $\omega = wdt$ for some function $w \in F$ that depends on our choice of $t$; note that $t$ is necessarily transcendental over $k$, since it is a uniformizer. We then define ord$_P(\omega) :=$ ord$_P(w)$, and the divisor of $\omega$ is then given by

$$\text{div} \omega := \sum_P \text{ord}_P(\omega)P.$$ 

As in Problem 1, the value ord$_P(\omega)$ does not depend on the choice of the uniformizer $t$.

(h) Prove that div $udv = \text{div} u + \text{div} dv$ for any $u, v \in F$. Conclude that the set of nonzero differentials in $\Delta$ constitutes a linear equivalence class of divisors.

(i) Let $F = k(t)$ be the rational function field. Compute div $dt$ and prove that it is a canonical divisor. Conclude that a divisor $D \in \text{Div}_k C$ is canonical if and only if $D = \text{div} df$ for some transcendental $f \in F$.

Part (i) holds for arbitrary curves, but you are not asked to prove this. It follows that the space of differentials $\Delta$ plays the same role as the space of Weil differentials $\Omega$, and it has the virtue of making explicit computations much easier.

(j) Prove that the curve $x^2 + y^2 + z^2$ over $\mathbb{Q}$ has genus 0 (even though it is not isomorphic to $\mathbb{P}^1$ because it has no rational points) by explicitly computing a canonical divisor.

Problem 3. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = “mind-numbing,” 10 = “mind-blowing”), and how difficult you found the problem (1 = “trivial,” 10 = “brutal”). Also estimate the amount of time you spent on each problem.

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<th>Interest</th>
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Please rate each of the following lectures that you attended, according to the quality of the material (1 = “useless”, 10 = “fascinating”), the quality of the presentation (1 = “epic fail”, 10 = “perfection”), the pace (1 = “way too slow”, 10 = “way too fast”), and the novelty of the material (1 = “old hat”, 10 = “all new”).

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Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.
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