These problems are related to the material covered in Lectures 22-23. I have made every effort to proof-read them, but some errors may remain. The first person to spot each error will receive 1-5 points of extra credit.

The problem set is due by the start of class on 12/10/2013 and should be submitted electronically as a pdf-file e-mailed to the instructor. You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions and to identify collaborators and any sources not listed in the syllabus.

As usual, a curve is a smooth projective (irreducible) variety of dimension one.

**Problem 1. A genus 1 curve with no rational points (30 points)**

Consider the homogeneous polynomial

\[ f(x, y, z) = x^3 + 2y^3 + 4z^3. \]

(a) Prove that the zero locus of \( f \) is a plane curve \( C/\mathbb{Q} \).

(b) Prove that \( C \) has genus one.

(c) Prove that \( C \) has no \( \mathbb{Q} \)-rational points (so it is not an elliptic curve over \( \mathbb{Q} \)).

**Problem 2. Hyperelliptic curves (70 points)**

A hyperelliptic curve \( C/k \) is a curve of genus \( g \geq 2 \) whose function field is a separable quadratic extension of the rational function field \( k(x) \). The non-trivial element of \( \text{Gal}(k(C)/k(x)) \) is called the hyperelliptic involution. In this problem we consider hyperelliptic curves over a perfect field \( k \) whose characteristic is not 2 (so every quadratic extension of \( k(x) \) is separable).

(a) Let \( C/k \) be a hyperelliptic curve of genus \( g \). Prove that \( C \) can be defined by an affine equation of the form \( y^2 = f(x) \), where \( f \in k[x] \) is a polynomial of degree \( 2g + 1 \) or \( 2g + 2 \) (so \( C \) is the desingularization of the projective closure of this affine variety). (hint: consider the Riemann-Roch spaces \( \mathcal{L}(nD) \) where \( D \) is the pole divisor of \( x \), and proceed along the lines of the first part of the proof of Theorem 23.3; as a first step, figure out what the degree of \( D \) must be).

(b) Prove that the polynomial \( f \) in part (a) can be made squarefree, and that \( y^2 - f(x) \) is irreducible in \( k[x, y] \). Then show that if \( k \) is algebraically closed one can make \( f \) monic and of degree \( 2g + 1 \).

(c) Let \( f \) be any squarefree polynomial in \( k[x] \) of degree \( d \geq 5 \). Prove that the curve defined by \( y^2 = f(x) \) is a hyperelliptic curve of genus \( g \leq (d - 1)/2 \).

(d) Let \( C/k \) be a hyperelliptic curve of genus \( g \) defined by \( y^2 = f(x) \) with \( f \) squarefree of degree \( d \), where \( k \) is algebraically closed. Prove that there are at least \( d \) distinct places of \( k(C) \) that are fixed by the hyperelliptic involution, but not every place of \( k(C) \) is fixed by the hyperelliptic involution.
(e) Let $C/k$ be a function field of genus $g$ over an algebraically closed field $k$, and let $\sigma$ be an automorphism of $k(C)$ that fixes $k$. Prove that if $\sigma$ does not fix every place of $k(C)$ then it fixes at most $2g + 2$ places. (hint: show that there is a nonconstant function $x \in \mathcal{L}((g+1)P)$, where $P$ is a place not fixed by $\sigma$, and then show that every place fixed by $\sigma$ corresponds to a zero of $\sigma(x) - x$).

(f) Using (b), (c), and (d), prove that every equation of the form $y^2 = f(x)$ with $f \in k[x]$ a squarefree polynomial of degree $d \geq 5$ defines a hyperelliptic curve $C/k$ of genus $g = \lceil \frac{d-1}{2} \rceil$. Your proof should work whether or not $k$ is algebraically closed.

(g) Prove that every curve of genus 2 is hyperelliptic (hint: first show there exists an effective canonical divisor $W$, then consider a non-constant $x \in \mathcal{L}(W)$).

Problem 3. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = “mind-numbing,” 10 = “mind-blowing”), and how difficult you found the problem (1 = “trivial,” 10 = “brutal”). Also estimate the amount of time you spent on each problem.

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<thead>
<tr>
<th>Problem</th>
<th>Interest</th>
<th>Difficulty</th>
<th>Time Spent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Problem 2</td>
<td></td>
<td></td>
<td></td>
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Please rate each of the following lectures that you attended, according to the quality of the material (1=“useless”, 10=“fascinating”), the quality of the presentation (1=“epic fail”, 10=“perfection”), the pace (1=“way too slow”, 10=“way too fast”), and the novelty of the material (1=“old hat”, 10=“all new”).

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<thead>
<tr>
<th>Date</th>
<th>Lecture Topic</th>
<th>Material</th>
<th>Presentation</th>
<th>Pace</th>
<th>Novelty</th>
</tr>
</thead>
<tbody>
<tr>
<td>11/26</td>
<td>Elliptic curves</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12/3</td>
<td>Isogenies and torsion points</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12/5</td>
<td>The Mordell-Weil theorem</td>
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Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.
18.782 Introduction to Arithmetic Geometry
Fall 2013

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