LECTURE 13

Homotopy Coinvariants, Abelianization, and Tate Cohomology

Recall that last time we explicitly constructed the homotopy invariants $X^h_G$ of a complex $X$ of $G$-modules. To do this, we constructed the bar resolution $P^\text{can}_G\xrightarrow{\partial} Z$, where $P^\text{can}_G$ is a canonical complex of free $G$-modules in non-positive degrees. Then we have a quasi-isomorphism $X^h_G \simeq \text{Hom}_G(P^\text{can}_G, X)$.

In particular, we have

$$\cdots \rightarrow Z[G^3] \xrightarrow{\partial} Z[G \times G] \xrightarrow{\partial} Z[G] \xrightarrow{\partial} 0 \xrightarrow{\partial} \cdots$$

for $P^\text{can}_G$, with differential of the form $(g_1, g_2) \mapsto g_1 g_2 - g_1$ (for $d^{-1}$; the $G$-action is always on the first term). Note that if $G$ is finite, then these are all finite-rank $G$-modules.

For every $G$-module $M$, we have

$$\cdots \rightarrow 0 \rightarrow M \xrightarrow{m \mapsto (gm - m)_{g \in G}} \prod_{g \in G} M \xrightarrow{\prod_{g,h \in G}} M \rightarrow \cdots$$

via some further differential, for $M^h_G$. We can use this expression to explicitly compute the first cohomology of $M^h_G$. It turns out that a function $\varphi : G \rightarrow M$ is killed by this differential if it is a 1-cocycle (sometimes called a twisted homomorphism), that is, $\varphi(gh) = \varphi(g) + g \cdot \varphi(h)$ for all $g, h \in G$ via the group action. Similarly, $\varphi$ is a 1-coboundary if there exists some $m \in M$ such that $\varphi(g) = g \cdot m - m$ for all $g \in G$. The upshot is that

$$H^1(G, M) := H^1(M^h_G) = \{1\text{-cocycles}\}/\{1\text{-coboundaries}\}.$$ 

As a corollary, if $G$ acts trivially on $M$, then $H^1(G, M) = \text{Hom}_{\text{Group}}(G, M)$, since the 1-coboundaries are all trivial, and the 1-cocycles are just ordinary group homomorphisms. This also shows that zeroth cohomology is just the invariants, as we showed last lecture.

Now, our objective (from a long time ago) is to define Tate cohomology and the Tate complex for any finite group $G$. We’d like $H^0(G, M) = M^G/N(M) = \text{Coker}(M_G \xrightarrow{N} M^G)$, because it generalizes the central problem of local class field theory for extensions of local fields. Recall that $M_G = M/(g - 1)$ (equivalent to tensoring with the trivial module, and dual to invariants, which we prefer as a submodule), so that this map factors through $M$ and induced the norm map above.
Our plan is, for a complex $X$ of $G$-modules, to form $h^G X \to X^{tG} := \text{hCoker}(N)$.

Thus, we first need to define the homotopy coinvariants $h^G X$. Note that if $M$ is a $G$-module, then $M^G = M \otimes_{\mathbb{Z}[G]} \mathbb{Z}$. Define $I^G := \text{Ker}(\epsilon)$, so that we have a short exact sequence

$$0 \to I^G \to \mathbb{Z}[G] \to Z \to 0$$

We claim that $I^G$ is $\mathbb{Z}$-spanned by $\{g-1 : g \in G\}$ (which we leave as an exercise).

A corollary is that $\mathbb{Z}[G]^G \to Z^G \to Z \to 0$ is exact, since $\mathbb{Z}[G]^G \to I^G$ via $1 \mapsto g-1$ on the $g$th coordinate.

Remark 13.1. The correct algorithm for computing tensor products is as follows: recall that tensor products are right-exact, that is, they preserve surjections, and tensoring with the algebra gives the original module. To tensor with a module, take generators and relations for that module, use it to write a resolution as above, tensor with that resolution, giving a matrix over a direct sum of copies of that module, and then take the cokernel.

It would be very convenient if we could define $M^hG$ via an analogous process for chain complexes.

Definition 13.2. If $X$ and $Y$ are chain complexes, then $(X \otimes Y)^i := \bigoplus_{j \in \mathbb{Z}} X^j \otimes Y^{i-j}$, with differential $d(x \otimes y) := dx \otimes y + (-1)^j x \otimes dy$

If $X$ is a complex of right $A$-modules, and $Y$ is a complex of left $A$-modules, then $X \otimes_A Y$ is defined similarly.

Note that the factor of $(-1)^j$ ensures that the differential squares to zero. Also, there is no need to worry about left and right $A$-modules for algebras, since left and right algebras are isomorphic via changing the order of multiplication; for $G$-modules, this means replacing every element with its inverse.

Now, a bad guess for $h^G X$ would be $X \otimes_{\mathbb{Z}[G]} \mathbb{Z}$, because it doesn’t preserve acyclic complexes, equivalently quasi-isomorphisms. A better guess is to take a projective resolution $P_G \simeq \mathbb{Z}$, e.g. $P^G_{\text{an}}$, and tensor with that instead: $h^G X := X \otimes_{\mathbb{Z}[G]} P_G$.

Definition 13.3. A complex $F$ of left $A$-modules is flat is for every acyclic complex $Y$ of right $A$-modules, $Y \otimes_A F$ is also acyclic, that is, $- \otimes_A F$ preserves injections.

We now ask if $P_G$ is flat. In fact:

Claim 13.4. Any projective complex is flat.

An easier claim is the following:

Claim 13.5. Any complex $F$ that is bounded above with $F^i$ flat for all $i$ is flat.
To prove this claim, we will use the fact that projective modules are flat, as they
are direct summands of free modules, which are trivially flat (i.e., if $F = F_1 \oplus F_2$,
then $F \otimes M = (F_1 \otimes M) \oplus (F_2 \otimes M)$).

**Proof.** Case 1. Suppose $F$ is in degree 0 only, i.e., $F^i = 0$ for all $i \neq 0$. For
every complex $Y = Y^\bullet$, we have

$$
\cdots \to Y^i \otimes_A F \xrightarrow{d^i \otimes \text{id}} Y^{i+1} \otimes_A F \to \cdots
$$

for $Y \otimes_A F$. Since $F$ is flat, we have $H^i(Y \otimes_A F) = H^i(Y) \otimes_A F$ for each $i$ (since
$F$ flat means that tensoring with $F$ commutes with forming kernels, cokernels and
images), so if $Y$ is acyclic, then $Y \otimes_A F$ is as well.

Case 2. Suppose $F$ is in degrees 0 and $-1$ only, i.e., $F$ is of the form

$$
\cdots \to 0 \to F^{-1} \to F^0 \to 0 \to \cdots,
$$

and so $F^\bullet = h\text{Coker}(F^{-1} \to F^0)$. Then since tensor products commute
with homotopy cokernels, we obtain

$$
Y \otimes_A F = h\text{Coker}(Y \otimes_A F^{-1} \to Y \otimes_A F^0),
$$

so by Case 1, if $Y$ is acyclic, then $Y \otimes_A F^0$ and $Y \otimes_A F^{-1}$ are as well, hence
$Y \otimes_A F$ is as well by the long exact sequence on cohomology. A similar (inductive)
argument gives the case where $F$ is bounded.

Case 3. In the general case, form the diagram

$$
\begin{array}{cccccccc}
F_0 & \to & 0 & \to & 0 & \to & 0 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
F_1 & \to & 0 & \to & 0 & \to & F^1 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
F_2 & \to & 0 & \to & F^2 & \to & F^1 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \\
\end{array}
$$

Clearly all squares of this diagram commute, hence these are all morphisms
of complexes, and $F = \lim_{\longrightarrow} F_i$. Since direct limits commute with tensor products
(note that is not true for inverse limits because of surjectivity), we have $Y \otimes_A F = \lim_{\longrightarrow} Y \otimes_A F_i$. By Case 2, $Y \otimes_A F_i$ is acyclic for each $i$, so since cohomology commutes
with direct limits (because they preserve kernels, cokernels, and images), if $Y$ is
acyclic, then $Y \otimes_A F$ is too.

**Remark 13.6.** Let $Y$ be a complex of $A$-modules, choose a quasi-isomorphism
$F \xrightarrow{\text{dis}} Y$, where $F$ is flat, and define $Y \otimes_A^{\text{der}} X := F \otimes_A X$. Then this is well-defined
up to quasi-isomorphism, which is well-defined up to homotopy, etc. (it’s turtles
all the way down!!)

**Definition 13.7.** The $i$th torsion group (of $Y$ against $X$) is $\text{Tor}^A_i(Y, X) := H^{-i}(Y \otimes_A^{\text{der}} X)$.

**Definition 13.8.** The homotopy coinvariants of a chain complex $X$ is the
complex $Xh_G := X \otimes_G^{\text{der}} \mathbb{Z} \simeq X \otimes_{\mathbb{Z}[G]} P_G$ (which we note is only well-defined up
to quasi-isomorphism).
Definition 13.9. $H_i(G, X) := H^{-i}(X_{hG})$ (where we note that the subscript notation is preferred as $X_{hG}$ is generally a complex in non-positive degrees only).

We now perform some basic calculations.

Claim 13.10. If $X$ is bounded from above by 0, then $H_0(G, X) = H^0(X)_G$ (the proof is similar to that of Claim 12.5).

Claim 13.11. $H_1(G, \mathbb{Z}) = G^{ab}$, where $G^{ab}$ denotes the abelianization of $G$.

Note that this is sort of a dual statement to what we saw at the beginning of lecture; $H_1(G, M)$ had to do with maps $G \to M$, which are the same as maps from $G^{ab} \to M$, and here $H_1(G, \mathbb{Z})$ is determined by the maps out of $G$.

Proof. Recall the short exact sequence

$0 \to I_G \to \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0$.

The long exact sequence on cohomology gives an exact sequence

$H_1(G, \mathbb{Z}[G]) \to H_1(G, \mathbb{Z}) \to H_0(G, I_G) \to H_0(G, \mathbb{Z}[G]) \to H_0(G, \mathbb{Z})$.

We have

$H_0(G, \mathbb{Z}[G]) = H^0(\mathbb{Z}[G])_G = \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} \mathbb{Z} = \mathbb{Z}$

by Claim 13.10. Certainly $H_0(G, \mathbb{Z}) = H^0(\mathbb{Z})_G = \mathbb{Z}$, and $H_1(G, \mathbb{Z}[G]) = 0$ as

$\mathbb{Z}[G]_{hG} := \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} P_G = P_G \cong \mathbb{Z}$

is a quasi-isomorphism. Thus, our exact sequence is really

$0 \to H_1(G, \mathbb{Z}) \xrightarrow{\sim} H_0(G, I_G) \to \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$,

which gives the noted isomorphism. The upshot is that

$H_1(G, \mathbb{Z}) = (I_G)_G = I_G/I_G^2$

since $M_G = M/I_G \cdot M$.

Claim 13.12. The map

$\mathbb{Z}[G]/I_G^2 \to G^{ab} \times \mathbb{Z}, \quad g \mapsto (\bar{g}, 1)$

is an isomorphism.

This would imply that $I_G/I_G^2 = \text{Ker}(\epsilon)/I_G^2 = G^{ab}$, as desired.

Proof. First note that the map above is a homomorphism. Indeed, letting $[g] \in \mathbb{Z}[G]$ denote the class of $g$, we have

$[g] + [h] \mapsto (\bar{g}\bar{h}, 2)$

$[g] \mapsto (\bar{g}, 1)$

$[h] \mapsto (\bar{h}, 1)$

for any $g, h \in G$, and the latter two images add up to the first. We claim that this map has an inverse, induced by the map

$G \times \mathbb{Z} \to \mathbb{Z}[G]/I_G^2, \quad (g, n) \mapsto [g] + n - 1$.

This is a homomorphism, as

$([g] - 1)([h] - 1) = [gh] - [g] - [h] + 1 \in I_G^2$. 

and therefore
\((g-1) + (h-1) \equiv [gh] - 1 \mod I_G^2,\)
as desired. Finally, they are inverses, as
\((\bar{g}, 1) \mapsto [g] + 1 - 1 = [g] \quad \text{and} \quad [g] + n - 1 \mapsto (\bar{g}, 1)(1, n - 1) = (\bar{g}, n),\)
as desired. \qed

This proves the claim. \qed

Finally, we define the norm map \(X_{hG} \to X^hG\) to be the composition
\[ X_{hG} = X \otimes_{\mathbb{Z}[G]} P_G \to X \otimes_{\mathbb{Z}[G]} \mathbb{Z} \to \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, X) \to \text{Hom}_{\mathbb{Z}[G]}(P_G, X) = X^hG, \]
where the second map is via degree-wise norm maps (using tensor-hom adjunction).
We then set
\[ X^{tG} := \text{hCoker}(X_{hG} \to X^hG), \]
which we claim generalizes what we had previously for cyclic groups up to quasi-isomorphism, so that we may define
\[ \hat{H}^i(G, X) := H^i(X^{tG}). \]

Soon we will prove:

**Claim 13.13 (lcft).** For a finite group \(G\) and extension \(L/K\) of local fields,
\[ P_G \to L^\times[2] \]
is an isomorphism on Tate cohomology.

This gives that
\[ \hat{H}^{-2}(G, \mathbb{Z}) \simeq \hat{H}^0(G, L^\times) = K^\times/N(L^\times). \]

We have an exact sequence
\[ 0 = \hat{H}^{-2}(\mathbb{Z}^hG) \to \hat{H}^{-2}(G, \mathbb{Z}) \xrightarrow{\sim} \hat{H}^{-1}(\mathbb{Z}_{hG}) \to H^{-1}(\mathbb{Z}^hG) = 0, \]
where \(H_i(G, \mathbb{Z}) = G^{ab}\)
since \(\mathbb{Z}^hG\) is in non-negative degrees. Thus, for an extension \(L/K\) of local fields with Galois group \(G\), we have
\[ L^\times/N(L^\times) \simeq G^{ab}. \]
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