The Vanishing Theorem Implies Cohomological LCFT

Last time, we reformulated our problem as showing that, for an extension $L/K$ of nonarchimedean local fields with Galois group $G$,

$$ (L^\times)^G \simeq \mathbb{Z}^G[-2]. $$

Thus, our new goal is to compute the Tate cohomology of $L^\times$. Recall that we have let $K^{\text{unr}}$ denote the completion of the maximal unramified extension of $K$; we’d like to use $K^{\text{unr}}$ to compute this Tate cohomology.

**Claim 15.1.** If $x \in K^{\text{unr}}$ is algebraic over $K$ (which may not be the case due to completion), then $K' := K(x)$ is unramified over $K$.

**Proof.** As a finite algebraic extension of $K$, $K'$ is a local field, and we have an embedding $O_{K'}/\mathfrak{p}_{K'}O_{K'} \hookrightarrow O_{K^{\text{unr}}}/\mathfrak{p}_{K}O_{K^{\text{unr}}} = \overline{k}$, where $k := O_K/\mathfrak{p}_K$. So $O_{K'}/\mathfrak{p}_{K}O_{K'}$ is a field, hence uniformizers of $K$ and $K'$ are identical. $\square$

**Claim 15.2.** $(K^{\text{unr}})^{\sigma=1} = K$, that is, the elements fixed by (i.e., on which it acts as the identity) the Frobenius automorphism $\sigma \in G$ (obtained from the Frobenii of each unramified extension, passed to the completion via continuity).

Recall that we have a short exact sequence

$$ 0 \rightarrow K \rightarrow K^{\text{unr}} \xrightarrow{1-\sigma} K^{\text{unr}}, $$

which we may rewrite on multiplicative groups as

$$ 1 \rightarrow K^\times \rightarrow K^{\text{unr},\times} \xrightarrow{x \mapsto x/\sigma x} K^{\text{unr},\times} \xrightarrow{v} \mathbb{Z} \rightarrow 0. $$

We showed that an element of $K^{\text{unr},\times}$ can only be written as $x/\sigma x$ if it is a unit in the ring of integers $O_{K^{\text{unr},\times}}$; this map is an isomorphism on each of the associated graded terms, hence on $O_{K^{\text{unr},\times}}$.

Now, we’d like to explicitly construct the isomorphism in (15.1). Our first attempt is as follows: let us assume that $L/K$ is totally ramified (since we discussed the unramified case last time, this is a rather mild assumption), so that $L^{\text{unr}} = L \otimes_K K^{\text{unr}}$. Then we have the following theorem, to be proved later.

**Theorem 15.3 (Vanishing Theorem).** If $L/K$ is totally ramified, then the complex $(L^{\text{unr},\times})^G$ is acyclic.

**Claim 15.4.** The vanishing theorem implies cohomological LCFT.
Proof. Assume $L/K$ is totally ramified. We have the four-term exact sequence

\begin{equation}
1 \to \mathcal{L} \to \mathcal{L} \times \sigma \mathcal{L} \to \mathcal{L} \times 0.
\end{equation}

We may rewrite this as follows:

\[
\begin{array}{cccccc}
A & \cdots & \to & 0 & \longrightarrow & L^x \\
\downarrow & & & \downarrow & & \downarrow \\
B & \cdots & \to & 0 & \longrightarrow & \mathcal{L} \times \mathcal{L} \\
\downarrow & & & \downarrow & & \downarrow \\
Coker(A \to B) & \cdots & \to & 0 & \longrightarrow & 0
\end{array}
\]

where $L^x$ is in degree $-1$. The final quasi-isomorphism to the homotopy cokernel obtained from (15.2) follows from Claim 10.12, because $A \hookrightarrow B$ is an injection (note that this holds in general for any four-term exact sequence). The term-wise cokernel yields an injection

\[
\mathcal{L} \times L^x / \mathcal{L} \times \sigma \mathcal{L} \to \mathcal{L} \times \mathcal{L}
\]

since, omitting the quotient, $L^x$ is precisely the kernel of this map.

Now, we have a quasi-isomorphism

\[
B^G = \text{hCoker}(\mathcal{L} \times (L^x \times \mathcal{L}))^G \simeq \text{hCoker}(\mathcal{L}^G \to (\mathcal{L} \times \mathcal{L}))^G,
\]

so since $(\mathcal{L} \times \mathcal{L})^G$ is acyclic by the vanishing theorem, this homotopy cokernel is as well by the long exact sequence on cohomology. Thus,

\[
(L^x[2])^G = \text{hCoker}((L^x[1])^G \to 0) = \text{hCoker}(A^G \to B^G) \simeq \mathbb{Z}^G,
\]

as desired. \qed

Now suppose $L/K$ is a general finite Galois extension with $G := \text{Gal}(L/K)$ (though we could handle the unramified and totally ramified cases separately, as any extension is canonically a composition of such extensions). If $L/K$ is unramified, then

\[
L \otimes_K \mathcal{K} = \prod_{L \hookrightarrow \mathcal{K}} \mathcal{K}
\]

canonicaly, indexed by such embeddings. In fact, the following holds:

**Theorem 15.5 (General Vanishing Theorem).** $[(L \otimes_K \mathcal{K})^*]^G$ is acyclic.

To understand the structure of $L \otimes_K \mathcal{K}$, note that we have an action of $\mathbb{Z}[\sigma]$ on the second factor and of $G$ on the first; these two actions (i.e., $x \otimes y \mapsto gx \otimes y$ and $x \otimes y \mapsto x \otimes sy$) clearly commute. Again, the points fixed under $\sigma$ are $L = L \otimes_K \mathcal{K}

\[
\text{Claim 15.6. } The following sequence is exact:}
\]

\[
1 \to \mathcal{L} \to (L \otimes_K \mathcal{K})^x \to \sigma \to 0.
\]

**Proof.** If $x \in \mathcal{K}$ is a unit, then $\sigma x$ is as well, so the map $x \mapsto x/\sigma x$ is well-defined, and moreover, $x$ is in its kernel if and only if $x$ is fixed under the
action of $\sigma$, that is, $x \in K$, and since $L \otimes_K K = L$ we obtain a unit of $L$, which shows exactness of the left half. Now, the map to $\mathbb{Z}$ is defined by

\[
\begin{array}{ccc}
(L \otimes_K K^{\text{unr}}) \times & \longrightarrow & \mathbb{Z} \\
\downarrow & & \\
K \otimes_K K^{\text{unr}, \times} & & \mathbb{Z}
\end{array}
\]

where

\[N_{L/K}(x) := \prod_{g \in G} gx.\]

Thus, its kernel is $\mathcal{O}_K^{\times}$, which is precisely the image of $x \mapsto x/\sigma x$. Moreover, the map is surjective as $1 \otimes \pi \mapsto 1$.

Observe that if $L/K$ is totally ramified, then this is just our extension from before. Indeed, if we write $L^{\text{unr}} = L \otimes_K K^{\text{unr}}$, then the $\sigma$’s “match up,” that is, the induced Frobenius automorphisms of $L^{\text{unr}}$ and $K^{\text{unr}}$ are identical as $L$ and $K$ have the same residue field. The norm $N_{L/K} : L^{\text{unr}, \times} \to K^{\text{unr}, \times}$ for this extension satisfies $v_{K^{\text{unr}}} \circ N = v_{L^{\text{unr}}}$ (such an extension is generated by the $n$th root of a uniformizer of $K$, and then $N(\pi^{1/n}) = \pi$).

Now suppose $L/K$ is unramified of degree $n$. Fix an embedding $L \hookrightarrow K^{\text{unr}}$, and let $\sigma \in \text{Gal}(L/K)$ also denote the Frobenius element of $L/K$. Then we have an isomorphism

\[L \otimes_K K^{\text{unr}} \simeq \prod_{i=0}^{n-1} K^{\text{unr}}\]

where the product is taken via our fixed embedding (note that this could be done more canonically by taking the product over embeddings as before). We now ask: what does the automorphism $\text{id} \otimes \sigma$ of $L \otimes_K K^{\text{unr}}$ correspond to under this isomorphism? We have

\[x \otimes y \mapsto (x \cdot \sigma y, x \cdot \sigma^2 y, \ldots, x \cdot \sigma^n y),\]

so it is the action of $\sigma$ on the rotation to the right of the image of $x \otimes y$ (note that $\sigma$ doesn’t have finite order on $K^{\text{unr}}$, so this should either, which rules our rotation as a possibility for the image of $\text{id} \otimes \sigma$). Similarly, the norm $N_{L/K} : \prod K^{\text{unr}, \times} \to K^{\text{unr}, \times}$ takes the product of all entries.

We’d like for some element $(x_0, \ldots, x_{n-1}) \in \prod K^{\text{unr}, \times}$ to be in the image of $y/\sigma y$ (i.e., the map in the middle of the exact sequence of Claim 15.6; here $\sigma$ refers to the automorphism $\text{id} \otimes \sigma$) if and only if $\prod x_i \in \mathcal{O}_K^{\times}$, that is, $\sum v(x_i) = 0$. Recall that the reverse implication is trivial, as we have shown that $\mathcal{O}_K^{\times} \xrightarrow{y/\sigma y} \mathcal{O}_K^{\times}$ is surjective as it is at the associated graded level. For the forward direction, we have

\[(y_0, \ldots, y_{n-1}) \xrightarrow{y/\sigma y} \left( \frac{y_0}{\sigma y_{n-1}}, \frac{y_1}{\sigma y_0}, \ldots \right) =: (x_0, x_1, \ldots).\]

Thus,

\[y_0 = x_0 \cdot \sigma y_{n-1},\]
\[ y_1 = x_1 \cdot \sigma y_0 = x_1 \cdot \sigma x_0 \cdots \sigma^2 y_{n-1}, \]
\[ \cdots = \cdots \]
\[ y_{n-1} = x_{n-1} \cdot \sigma x_{n-2} \cdots \sigma^{n-1} x_0 \cdot \sigma^n y_{n-1}, \]

that is,
\[ \frac{y_{n-1}}{\sigma y_{n-1}} = x_{n-1} \cdot \sigma x_{n-2} \cdots \sigma^{n-1} x_0. \]

Note that everything here is an element of \( K^\text{unr} \), so we really do not have \( \sigma^n = \text{id} \)!

Last time, we showed that we can do this if and only if the right-hand side is in \( O_{K^\text{unr}}^x \), which is equivalent to saying that \( \sum v(x_i) = 0 \). The general case of this exact sequence is sort of a mix of the two.

We now compare these results with those from the last lecture. Assume the Vanishing Theorem. For an unramified extension \( L/K \), we have two quasi-isomorphisms between \( (L^x)^G \) and \( Z[-2]^G \), one from what we just did, and the other since \( (O_L^x)^G \simeq 0 \) implies \( (L^x)^G \simeq Z^G \simeq (Z[-2])^G \) by cyclicity. We claim that these two quasi-isomorphisms coincide. A sketch of the proof is as follows: we have \( G = Z/nZ \) (with generator the Frobenius element), and a short exact sequence
\[ 0 \rightarrow Z \rightarrow Z[G] \rightarrow \sum_v Z \rightarrow 0. \]

As shown in Problem 1(e) of Problem Set 7, \( Z[G]^G \simeq 0 \) is a quasi-isomorphism (this is easy to show, and we’ve already shown it for cyclic groups). Thus, we get \( Z[G][2] \simeq Z^G \), and we claim that this is the same isomorphism that we get from 2-periodicity of the complex. The proof is by a diagram chase. We have \( (L \otimes_K K^\text{unr})^x = \prod K^\text{unr},^x \), which is a finite product. Thus, the diagram
\[
\begin{array}{ccc}
1 & \longrightarrow & L^x \\
\downarrow v & & \downarrow \prod_v (L \otimes_K K^\text{unr})^x \\
1 & \longrightarrow & \prod_{i=0}^n Z \\
\end{array}
\]

\[
\begin{array}{ccc}
\sum_v Z & \longrightarrow & 0 \\
\downarrow \prod_v & & \downarrow \prod_v \\
\end{array}
\]

\[
\begin{array}{ccc}
Z[G] \simeq \prod_v Z & \longrightarrow & Z \\
\downarrow \epsilon & & \downarrow \epsilon \\
Z & \longrightarrow & 0
\end{array}
\]

commutes, where \( \epsilon \) denotes the sum over the coordinates of \( Z[G] \). This says precisely that the isomorphisms obtained from both 4-term exact sequences coincide.

The upshot is that under \text{lcft}, we have an isomorphism \( K^x/NL^x \simeq Z/nZ \) by which \( \pi \mapsto \text{Frob} \). Thus, we have reduced \text{lcft} to the Vanishing Theorem, which we will prove in the next lecture.