4. Let $K$ be a finite extension of $\mathbb{Q}_p$. Let $f(x) = x^n + \sum_{i=0}^{n-1} f_i x^i$ be a monic polynomial of degree $n$ over $K$, which factors completely over $K$ with distinct roots $r_1, \ldots, r_n$. Prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $g(x) = x^n + \sum_{i=0}^{n-1} g_i x^i$ is a monic polynomial of degree $n$ such that $|f_i - g_i| < \delta$, then $g$ has $n$ roots $s_1, \ldots, s_n$ in $K$, which can be labeled so that $|r_i - s_i| < \epsilon$ for $i = 1, \ldots, n$. That is, the roots of $f$ vary continuously with the coefficients. (If the roots are not distinct, the roots of $g$ may only lie in an extension of $K$, but otherwise the conclusion still holds.)
5. Let $K$ be a finite extension of $\mathbb{Q}_p$. Prove Krasner’s Lemma: if $\alpha_1, \ldots, \alpha_n \in \overline{K}$ are conjugates, and $\beta \in K$ satisfies

$$|\alpha_1 - \beta| < |\alpha_1 - \alpha_i| \quad (i = 2, \ldots, n),$$

then $K(\alpha_1) \subseteq K(\beta)$.
6. (Abhyankar’s Lemma) Let $K$ be a finite extension of $\mathbb{Q}_p$. A finite extension $L/K$ is said to be tamely ramified if $e(\mathfrak{m}_L/\mathfrak{m}_K)$ is coprime to $p$. Let $L_1, L_2$ be tamely ramified extensions of $K$ such that $e(\mathfrak{m}_{L_1}/\mathfrak{m}_K)$ divides $e(\mathfrak{m}_{L_2}/\mathfrak{m}_K)$. Prove that the compositum $L_1L_2$ is unramified over $L_2$. (Hint: it is safe to check this after making an unramified extension of $K$, so you can assume $L_1$ and $L_2$ are both Kummer extensions.)
7. (Dwork) Let $p$ be a prime number. Show that $\mathbb{Q}_p(\zeta_p) = \mathbb{Q}_p(\pi)$ for $\pi$ a $(p-1)$-st root of $-p$. (Hint: either of the previous two exercises might be helpful, or you can explicitly construct a series in $\pi$ converging to $\zeta_p$.)
8. (Optional because it uses some topology, but strongly recommended) Let $K$ be a number field. Let $\mathbb{A}_K$ be the subring of the product $\prod_v K_v$, where $v$ runs over places and $K_v$ is the completion at $v$, consisting of tuples $(a_v)$ in which $a_v \in \mathfrak{o}_{K_v}$ for all but finitely many finite places $v$ (no condition is imposed at infinite places). Give $\mathbb{A}_K$ the topology with a basis of open sets given by products $\prod_v U_v$, with $U_v$ open in $K_v$ and $U_v = \mathfrak{o}_{K_v}$ for all but finitely many finite $v$. Prove that $K$, which naturally embeds into $\mathbb{A}_K$ via the maps $K \hookrightarrow K_v$, is a discrete subgroup of $\mathbb{A}_K$ and that the quotient $\mathbb{A}_K/K$ is compact; that is, in some sense $K$ is a “full lattice” in $\mathbb{A}_K$. (Hint: start with Tykhonov’s theorem that any product of compact spaces is compact.) The ring $\mathbb{A}_K$ is the ring of adèles of $K$; we’ll likely see it again later. (There’s a multiplicative analogue too; more on that later.)