18.786 Problem Set 9 (due Thursday Apr 22 in class)

1. Let $L/K$ be a finite extension of finite fields. Show that the norm from $L$ to $K$ is surjective. (Hint: use Hilbert’s theorem 90 and the Herbrand quotient.)

2. Let $L/K$ be a finite extension of finite fields. Show that the trace map from $L$ to $K$ is surjective.

3. Check that $M_G \cong M \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ where $\mathbb{Z}$ is considered as a trivial $\mathbb{Z}[G]$ module. (Hint: normal basis theorem).

   Note: for homology, the corestriction map is natural and defined as the map induced by defining $\text{Cor} : H_0(H, M) \to H_0(G, M)$ in dimension 0 as $M_H = M/I_H M \to M/I_G M = M_G$, noting that $I_H \subset I_G$, and extending to higher dimensions by using Shapiro’s lemma.
   On the other hand, restriction in dimension 0 is $M_G \to M_H$ given by $m \mapsto \sum_{s \in S} s^{-1} m$, where $G = \bigcup_{s \in S} sH$.
   (Hint: Consider the exact sequence of $G$ or $H$ modules $0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ and take homology with respect to $G$ and $H$ and compare).

5. Prove that the Galois group of a finite extension of local fields is solvable, as follows. Let $L/K$ be a finite Galois extension with Galois group $G$, with $v_K$ a discrete normalized valuation of $K$ which therefore admits a unique extension $w$ to $L$. Let $v_L = ew$ be the associated normalized valuation of $L$, where $e$ is the ramification index of $L/K$ (i.e. we want $v_K(\pi_K) = v_L(\pi_L) = 1$).
   For every real number $s \geq -1$ define the $s$th ramification group of $L/K$ by $G_s = \{ g \in G \mid v_L(ga - a) \geq s + 1 \ \forall a \in \mathcal{O}_L \}$.
   (a) Prove that the $G_s$ form a chain $G = G_{-1} \supset G_0 \supset G_1 \supset \ldots$ of normal subgroups of $G$.
   (b) Show $G_{-1}/G_0$ is cyclic.
   (c) For every integer $s \geq 0$, define the map $G_s/G_{s+1} \rightarrow U_L^{(s)}/U_L^{(s+1)}$ by sending $g$ to $g(\pi_L)/\pi_L$.
      (Here $U_L^{(s)} = 1 + m_L^s$ for $s \geq 1$ and $\mathcal{O}_L^*$ for $s = 0$.) Show that this is a well-defined injective homomorphism independent of the choice of uniformizer $\pi_L$.
   (d) Show that $G_s/G_{s+1}$ is a finite abelian group for every $s \geq 1$. Conclude that $G$ is solvable.