The Long Line

We follow the outline of Exercise 12 of §24.

Let $L$ denote the set $S_\Lambda \times [0,1)$, in the dictionary order. Let $\alpha_0$ denote the smallest element of $S_\Lambda$. Give $L$ the order topology.

**Lemma C.1.** Let $\alpha$ be a point of $S_\Lambda$ different from $\alpha_0$. Then the interval $[\alpha_0 \times 0, \alpha \times 0]$ of $L$ has the order type of $[0,1]$.

**Proof.** Note that the proof is trivial if $\alpha$ is the immediate successor of $\alpha_0$ in $S_\Lambda$.

Suppose the lemma holds for all $\alpha < \beta$. We show it holds for $\beta$.

If $\beta$ has an immediate predecessor $\gamma$, the proof is easy. The interval $[\alpha_0 \times 0, \gamma \times 0]$ of $L$ has the order type of $[0,1]$ by hypothesis.

The interval $[\gamma \times 0, \beta \times 0]$ of $L$ equals $(\cup [0,1) \cup \sum \beta \times 0)$, so it has the order type of $[0,1]$, and also of $[1,2]$. Their union has the order type of $[0,1] \cup [1,2] = [0,2]$, which of course has the order type of $[0,1]$.

If $\beta$ has no immediate predecessor, there is an increasing sequence $\alpha_1, \alpha_2, \ldots$ of points of $S$ whose supremum is $\beta$. Assume $\alpha_1 > \alpha_0$ for convenience. We show that for each $i$ the interval $[\alpha_i \times 0, \alpha_{i+1} \times 0]$ of $L$ has the order type of $[0,1]$. The interval $[\alpha_0 \times 0, \alpha_{i+1} \times 0]$ has the order type of $[0,1]$ by hypothesis; if $\alpha_0 \times 0$ corresponds to the real number $c$ of $[0,1]$ under the order-preserving bijection, then $[\alpha_i \times 0, \alpha_{i+1} \times 0]$ has the order type of $[c,1]$, which of course has the order type of $[0,1]$.

Finally, we note that the interval $J = [\alpha_0 \times 0, \beta \times 0]$ of $L$ can be written as the union

$$[\alpha_0 \times 0, \alpha_1 \times 0] \cup [\alpha_1 \times 0, \alpha_2 \times 0] \cup \ldots \cup [\alpha_i \times 0, \alpha_{i+1} \times 0] \cup \ldots$$

of intervals of $L$. There is an order-preserving correspondence of this union with the union

$$[0,1] \cup [1,2] \cup \ldots \cup [i, i+1] \cup \ldots$$

of intervals of $\mathbb{R}$. The latter union equals $[0, +\infty)$, which has the order type of $[0,1]$. When we adjoin the point $\beta \times 0$ to $J$, we obtain a set with the order type of $[0,1]$. □
Definition. Let $L'$ be the subspace $L - \{0\} \times 0$ of $L$; it is called the Long Line.

Theorem C.2. The long line is a path-connected linear continuum, every point of which has a neighborhood homeomorphic to an open interval of $\mathbb{R}$. It is not metrizable.

Proof. Let $x$ be a point of $L$ with $x \neq 0 \times 0$. Choose an element $\alpha$ of $S_\omega$ so that $x < \alpha \times 0$. Then $x$ lies in the open interval $(\alpha \times 0, \alpha \times 0)$ of $L$, which has the order type of the open interval $(0,1)$ of $\mathbb{R}$.

The fact that $L'$ is a linear continuum follows from Ex. 6 of §24. The result of the preceding paragraph shows that $L'$ is the union of the open intervals $(\alpha \times 0, \alpha \times 0)$ of $L$, each of which is path connected; since they have the point $\alpha \times \frac{1}{2}$ in common, $L'$ is path connected.

Now let $\alpha$ be the immediate successor of $\alpha_0$ in $S_\omega$. We show that the ray $R = [\alpha \times 0, +\omega)$ of $L'$ is limit point compact but not compact. It follows that $R$ is not metrizable, so neither is $L'$.

The fact that $R$ is not compact follows from the fact that the covering of $L'$ by the open sets $[\alpha \times 0, \beta \times 0]$ with $\beta > \alpha$ has no finite (or even countable) subcovering. To show $R$ is limit point compact, it suffices to show that every countably infinite set $S$ in $R$ has a limit point. And this is easy: The set of first coordinates of points of $S$ has an upper bound in $S_\omega$. If $\beta$ is the immediate successor of this upper bound, then $S$ is a subset of the interval $[\alpha \times 0, \beta \times 0]$ of $L'$. Since $L'$ is a linear continuum, this interval is compact; therefore $S$ has a limit point. □