Normality of Linear Continua

Theorem E.1. Every linear continuum $X$ is normal in the order topology.

Proof. It suffices to consider the case where $X$ has no largest element and no smallest element. For if $X$ has a smallest $x_0$ but no largest, we can form a new ordered set $Y$ by taking the disjoint union of $(C,1)$ and $X$, and declaring every element of $(0,1)$ to be less than every element of $X$. The ordered set $Y$ is a linear continuum with no largest or smallest. Since $X$ is a closed subspace of $Y$, normality of $Y$ implies normality of $X$.

The other cases are similar.

So suppose $X$ has no largest or smallest. We follow the outline of Exercise 8 of §32.

Step 1. Let $C$ be a nonempty closed subset of $X$. We show that each component of $X - C$ has the form $(c, +\infty)$ or $(-\infty, c)$ or $(c, c')$, where $c$ and $c'$ are points of $C$.

Given a point $x$ of $X - C$, let us take the union $U$ of all open intervals $(a_x, b_x)$ of $X$ that contain $x$ and lie in $X - C$. Then $U$ is connected.

We show that $U$ has one of the given forms, and that $U$ is one of the components of $X - C$.

Let $a = \inf a_x$ or $a = -\infty$, according as the set $\{a_x\}$ has a lower bound or not. Let $b = \sup b_x$ or $b = +\infty$ according as the set $\{b_x\}$ has an upper bound or not. Then $U = (a, b)$. If $a \neq -\infty$, we show $a$ is a point of $C$. Suppose that $a$ is not a point of $C$. Then there is an open interval $(d, e)$ about $a$ disjoint from $C$. This open interval contains $a_x$ for some $x$ because $a = \inf a_x$; then the union $(d, e) \cup (a_x, b_x)$ is an open interval that contains $x$ and lies in $X - C$. This contradicts the definition of $a$.

Similarly, if $b$ is not $+\infty$, then $b$ must be a point of $C$. We conclude that $U$ is of one of the specified forms. [The form $(-\infty, +\infty)$ is not possible, since $C$ is nonempty.]

It now follows that, because the end points of $U$ are $\pm \infty$ or in $C$, no larger subset of $X - C$ can be connected. Thus $U$ must be the component of $X - C$ that contains $x$. 

Step 2. Let $A$ and $B$ be disjoint closed sets in $X$. For each component $W$ of $X - A \cup B$ that is an open interval with one end point in $A$ and the other in $B$, choose a point $d_W$ in $W$. Let $D$ be the set of all the points $d_W$. We show that $D$ is closed and discrete.

We show that if $x$ is a limit point of $D$, then $x$ lies in both $A$ and $B$ (which is not possible). It follows that $D$ has no limit points.

We suppose that $x$ is not in $A$, and show that $x$ is not a limit point of $D$. Let $I$ be an open interval about $x$ that is disjoint from $A$; we show that $I$ contains at most two points of $D$. If $I$ contains the point $d_W$ of $D$, then $I$ intersects the corresponding set $W$, which has one of the forms $W = (a, b)$ or $W = (b, a)$, where $a \in A$ and $b \in B$. Because $I$ is disjoint from $A$, it can intersect at most one set of the form $W = (a,b)$ and at most one set of the form $W = (b,a)$.

---

Step 3. Let $V$ be a component of $X - D$. We show that $V$ cannot intersect both $A$ and $B$.

Suppose $V$ contains a point $a$ of $A$ and a point $b$ of $B$; assume for convenience that $a < b$. Being connected, $V$ must contain the interval $[a, b]$. Let $a_0$ be the supremum of the set $A \cap [a, b]$. Then $a_0$ lies in $A$ and $a_0 < b$. The set $(a_0, b]$ does not intersect $A$. Let $b_0$ be the infimum of the set $B \cap [a_0, b]$. Then $b_0$ lies in $B$ and $b_0 > a_0$. The interval $(a_0, b_0)$ contains no point of $A \cup B$; because its end points lie in $A \cup B$, no larger subset of $X - A \cup B$ can be connected. Hence $(a_0, b_0)$ is one of the components of $X - A \cup B$; as such, it contains a point of $D$. Hence $V$ contains a point of $D$, contrary to construction.

Step 4. By Step 1, the components of $X - D$ are open sets of $X$. Let $U_A$ be the union of all components of $X - D$ that intersect $A$, and let $U_B$ be the union of all components of $X - D$ that intersect $B$. Then $U_A$ and $U_B$ are disjoint open sets containing $A$ and $B$, respectively. □