1. Use naturality to show that if $X$ is the disjoint union of subspaces $X_\alpha$, then there is an isomorphism of rings

$$H^*(X) \rightarrow \prod_\alpha H^*(X_\alpha),$$

where the latter ring has the standard componentwise ring structure

$$(r_\alpha) \cdot (s_\alpha) = (r_\alpha s_\alpha).$$

2. Let $X$ be the lens space $L(n,1)$ from Hatcher, problem 8 on page 131. (We looked at this space on a previous assignment.) Compute (using the $\Delta$-complex structure when necessary) the cohomology and cup product structure on $H^*(X;\mathbb{Z})$ and $H^*(X;\mathbb{Z}/n)$.

3. Suppose that $C_*$ and $D_*$ are chain complexes. Define a new chain complex $\text{Hom}(C_*, D_*)$ which, in degree $n$, is

$$\text{Hom}(C_*, D_*)_n = \prod_p \text{Hom}(C_p, D_{p+n}).$$

In other words, an element of this chain complex in degree $n$ consists of a family of maps $f_p : C_p \rightarrow D_{p+n}$ (so $n$ is the amount by which each map raises degree).

Give a definition of a boundary map $\delta : \text{Hom}(C_*, D_*)_n \rightarrow \text{Hom}(C_*, D_*)_{n-1}$ such that the evaluation map

$$\text{Hom}(C_*, D_*) \otimes C_* \rightarrow D_*$$

$$f \otimes x \mapsto f(x)$$

is a chain map, where the left-hand side has the standard Leibniz formula for its boundary.

What does it mean for an element of $\text{Hom}(C_*, D_*)_0$ to be a cycle? When do two cycles in $\text{Hom}(C_*, D_*)_0$ differ by a boundary?

4. Show the following dual version of the Künneth formula: If $C_*$ is a chain complex where each $C_p$ is levelwise free, and $D_*$ is any chain complex, show that there are natural exact sequences

$$0 \rightarrow \prod_{t-s=n+1} \text{Ext}(H_s(C_*), H_t(D_*)) \rightarrow H_n(\text{Hom}(C_*, D_*)) \rightarrow \prod_{t-s=n} \text{Hom}(H_s(C_*), H_t(D_*)) \rightarrow 0$$

which are split. (Hint: Follow the same method as the proofs of the Künneth formula and the universal coefficient theorems.)