15 CW-complexes II

We have a few more general things to say about CW complexes.

Suppose $X$ is a CW complex, with skeleton filtration $\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X$ and cell structure

$$
\coprod_{\alpha \in A_n} S_{\alpha}^{n-1} \xrightarrow{f_n} X_{n-1},
\coprod_{\alpha \in A_n} D_{\alpha}^n \xrightarrow{g_n} X_n
$$

In each case, the boundary of a cell gets identified with part of the previous skeleton, but the "interior"

$$\text{Int} D^n = \{ x \in D^n : |x| < 1 \}$$

does not. (Note that $\text{Int} D^0 = D^0$.) Thus as sets – ignoring the topology –

$$X = \coprod_{n \geq 0} \coprod_{\alpha \in A_n} \text{Int}(D^n).$$

The subsets $\text{Int} D^n$ are called "open $n$-cells," despite the fact that they are not generally open in the topology on $X$, and (except when $n = 0$) they are not homeomorphic to compact disks.

**Definition 15.1.** Let $X$ be a CW-complex with a cell structure $\{ g_{n} : D_{\alpha}^n \to X_n : \alpha \in A_n, n \geq 0 \}$. A subcomplex is a subspace $Y \subseteq X$ such that for all $n$, there is a subset $B_n$ of $A_n$ such that $Y_n = Y \cap X_n$ provides $Y$ with a CW-structure with characteristic maps $\{ g_{\beta} : \beta \in B_n, n \geq 0 \}$.

**Example 15.2.** $\text{Sk}_n X \subseteq X$ is a subcomplex.

**Proposition 15.3.** Let $X$ be a CW-complex with a chosen cell structure. Any compact subspace of $X$ lies in some finite subcomplex.

*Proof. See [2], p. 196.*

**Remark 15.4.** For fixed cell structures, unions and intersections of subcomplexes are subcomplexes.

The $n$-sphere $S^n$ (for $n > 0$) admits a very simple CW structure: Let $\ast = \text{Sk}_0 (S^n) = \text{Sk}_1 (S^n) = \cdots = \text{Sk}_{n-1} (S^n)$, and attach an $n$-cell using the unique map $S^{n-1} \to \ast$. This is a minimal CW structure – you need at least two cells to build $S^n$.

This is great – much simpler than the simplest construction of $S^n$ as a simplicial complex – but it is not ideal for all applications. Here's another CW-structure on $S^n$. Regard $S^n \subseteq \mathbb{R}^{n+1}$, filter the Euclidean space by leading subspaces

$$\mathbb{R}^k = \langle e_1, \ldots, e_k \rangle.$$

and define

$$\text{Sk}_k S^n = S^n \cap \mathbb{R}^{k+1} = S^k.$$
Now there are two \( k \)-cells for each \( k \) with \( 0 \leq k \leq n \), given by the two hemispheres of \( S^k \). For each \( k \) there are two characteristic maps, 

\[ u, \ell : D^k \to S^k \]

defining the upper and lower hemispheres:

\[ u(x) = (x, \sqrt{1 - |x|^2}), \quad \ell(x) = (x, -\sqrt{1 - |x|^2}). \]

Note that if \( |x| = 1 \) then \( |u(x)| = |\ell(x)| = 1 \), so each characteristic map restricts on the boundary to a map to \( S^{k-1} \), and serves as an attaching map. This cell structure has the advantage that \( S^{n-1} \) is a subcomplex of \( S^n \).

The case \( n = \infty \) is allowed here. Then \( \mathbb{R}^\infty \) denotes the countably infinite dimensional inner product space that is the topological union of the leading subspaces \( \mathbb{R}^n \). The CW-complex \( S^\infty \) is of finite type but not finite dimensional. It has the following interesting property. We know that \( S^n \) is not contractible (because the identity map and a constant map have different behavior in homology), but:

**Proposition 15.5.** \( S^\infty \) is contractible.

**Proof.** This is an example of a “swindle,” making use of infinite dimensionality. Let \( T : \mathbb{R}^\infty \to \mathbb{R}^\infty \) send \((x_1, x_2, \ldots)\) to \((0, x_1, x_2, \ldots)\). This sends \( S^\infty \) to itself. The location of the leading nonzero entry is different for \( x \) and \( Tx \), so the line segment joining \( x \) to \( Tx \) doesn’t pass through the origin. Therefore

\[ x \mapsto \frac{tx + (1-t)Tx}{|tx + (1-t)Tx|} \]

provides a homotopy \( 1 \simeq T \). On the other hand, \( T \) is homotopic to the constant map with value \((1, 0, 0, \ldots)\), again by an affine homotopy. \( \square \)

This “inefficient” CW structure on \( S^n \) has a second advantage: it’s *equivariant* with respect to the antipodal involution. This provides us with a CW structure on the orbit space for this action.

Recall that \( \mathbb{R}P^k = S^k/\sim \) where \( x \sim -x \). The quotient map \( \pi : S^k \to \mathbb{R}P^k \) is a double cover, identifying upper and lower hemispheres. The inclusion of one sphere in the next is compatible with this equivalence relation, and gives us “linear” embeddings \( \mathbb{R}P^{k-1} \subseteq \mathbb{R}P^k \). This suggests that

\[ \emptyset \subseteq \mathbb{R}P^0 \subseteq \mathbb{R}P^1 \subseteq \cdots \subseteq \mathbb{R}P^n \]
might serve as a CW filtration. Indeed, for each $k$,

$$\begin{array}{c}
S^{k-1} \xrightarrow{} D^k \\
\downarrow \pi \quad \downarrow u \\
\text{RP}^{k-1} \xrightarrow{} \text{RP}^k
\end{array}$$

is a pushout: A line in $\mathbb{R}^{k+1}$ either lies in $\mathbb{R}^k$ or is determined by a unique point in the upper hemisphere of $S^k$. 
Bibliography


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