16 Homology of CW-complexes

The skeleton filtration of a CW complex leads to a long exact sequence in homology, showing that the relative homology $H_q(X_k, X_{k-1})$ controls how the homology changes when you pass from $X_{k-1}$ to $X_k$. What is this relative homology? If we pick a set of attaching maps, we get the following diagram.

\[
\begin{array}{ccc}
\coprod_{\alpha} S^{k-1}_{\alpha} & \xrightarrow{f} & \coprod_{\alpha} D^k_{\alpha} \\
\xrightarrow{f} & & \xrightarrow{f} \\
X_{k-1} & \xrightarrow{f} & X_k \cup_f B \xrightarrow{f} X_k/X_{k-1}
\end{array}
\]

where $\bigvee$ is the wedge sum (disjoint union with all basepoints identified): $\bigvee_{\alpha} S^k_{\alpha}$ is a bouquet of spheres. The dotted map exists and is easily seen to be a homeomorphism.

Luckily, the inclusion $X_{k-1} \subseteq X_k$ satisfies what’s needed to conclude that

\[ H_q(X_k, X_{k-1}) \rightarrow H_q(X_k/X_{k-1}, *) \]

is an isomorphism. After all, $X_{k-1}$ is a deformation retract of the space you get from $X_k$ by deleting the center of each $k$-cell.

We know $H_q(X_k/X_{k-1}, *)$ very well:

\[ H_q(\bigvee_{\alpha \in A_k} S^k_{\alpha}, *) \cong \begin{cases} 
\mathbb{Z}[A_k] & q = k \\
0 & q \neq k
\end{cases} \]

Lesson: The relative homology $H_k(X_k, X_{k-1})$ keeps track of the $k$-cells of $X$.

**Definition 16.1.** The group of *cellular $n$-chains* in a CW complex $X$ is

\[ C_k(X) := H_k(X_k, X_{k-1}) = \mathbb{Z}[A_k]. \]

If we put the fact that $H_q(X_k, X_{k-1}) = 0$ for $q \neq k, k+1$ into the homology long exact sequence of the pair, we find first that

\[ H_q(X_{k-1}) \cong H_q(X_k) \quad \text{for} \quad q \neq k, k-1, \]

and then that there is a short exact sequence

\[ 0 \rightarrow H_k(X_k) \rightarrow C_k(X) \rightarrow H_{k-1}(X_{k-1}) \rightarrow 0. \]

So if we fix a dimension $q$, and watch how $H_q$ varies as we move through the skelata of $X$, we find the following picture. Say $q > 0$. Since $X_0$ is discrete, $H_q(X_0) = 0$. Then $H_q(X_k)$ continues to
be 0 till you get up to \( X_q \). \( H_q(X_q) \) is a subgroup of the free abelian group \( C_q(X) \) and hence is free abelian. Relations may get introduced into it when we pass to \( X_{q+1} \); but thereafter all the maps

\[ H_q(X_{q+1}) \to H_q(X_{q+2}) \to \cdots \]

are isomorphisms. All the \( q \)-dimensional homology of \( X \) is created on \( X_q \), and all the relations in \( H_q(X) \) occur by \( X_{q+1} \).

This stable value of \( H_q(X_k) \) maps isomorphically to \( H_q(X) \), even if \( X \) is infinite dimensional. This is because the union of the images of any finite set of singular simplices in \( X \) is compact and so lies in a finite subcomplex and in particular lies in a finite skeleton. So any chain in \( X \) is the image of a chain in some skeleton. Since \( H_q(X_k) \xrightarrow{\cong} H_q(X_{k+1}) \) for \( k > q \), we find that \( H_q(X_q) \to H_q(X) \) is surjective. Similarly, if \( c \in S_q(X_k) \) is a boundary in \( X \), then it’s a boundary in \( X_\ell \) for some \( \ell \geq k \). This shows that the map \( H_q(X_{q+1}) \to H_q(X) \) is injective. We summarize:

**Proposition 16.2.** Let \( k, q \geq 0 \). Then

\[ H_q(X_k) = 0 \quad \text{for } k < q \]

and

\[ H_q(X_k) \xrightarrow{\cong} H_q(X) \quad \text{for } k > q. \]

In particular, \( H_q(X) = 0 \) if \( q \) exceeds the dimension of \( X \).

We have defined the cellular \( n \)-chains of a CW complex \( X \),

\[ C_n(X) = H_n(X_n, X_{n-1}), \]

and found that it is the free abelian group on the set of \( n \) cells. We claim that these abelian groups are related to each other; they form the groups in a chain complex.

What should the boundary of an \( n \)-cell be? It’s represented by a characteristic map \( D^n \to X_n \) whose boundary is the attaching map \( \alpha : S^{n-1} \to X_{n-1} \). This is a lot of information, and hard to interpret because \( X_{n-1} \) is itself potentially a complicated space. But things get much simpler if I pinch out \( X_{n-2} \). This suggests defining

\[ d : C_n(X) = H_n(X_n, X_{n-1}) \xrightarrow{\partial_n} H_{n-1}(X_n) \to H_{n-1}(X_{n-1}, X_{n-2}) = C_{n-1}(X). \]

The fact that \( d^2 = 0 \) is embedded in the following large diagram, in which the two columns and the central row are exact.

\[
\begin{array}{ccccccc}
C_{n+1}(X) = H_{n+1}(X_{n+1}, X_n) & \\
\downarrow \partial_{n+1} & & \downarrow d & & 0 = H_{n-1}(X_{n-2}) \\
H_n(X_n) & \xrightarrow{\partial_n} & C_n(X) = H_n(X_n, X_{n-1}) & \xrightarrow{\partial_{n-1}} & H_{n-1}(X_{n-1}) & \\
\downarrow j_n & & \downarrow d & & \downarrow j_{n-1} & \\
H_n(X_{n+1}) & & C_{n-1}(X) = H_{n-1}(X_{n-1}, X_{n-2}) & & \\
\downarrow & & \downarrow & & & \downarrow \\
0 = H_n(X_{n+1}, X_n) & & & & & \\
\end{array}
\]

Now, \( \partial_{n-1} \circ j_n = 0 \). So the composite of the diagonals is zero, i.e., \( d^2 = 0 \), and we have a chain complex! This is the “cellular chain complex” of \( X \).
We should compute the homology of this chain complex, $H_n(C_\ast(X)) = \ker d / \im d$. Now

$$\ker d = \ker(j_{n-1} \circ \partial_{n-1}).$$

But $j_{n-1}$ is injective, so

$$\ker d = \ker \partial_{n-1} = \im j_n = H_n(X_n).$$

On the other hand

$$\im d = j_n(\im \partial_n) = \im \partial_n \subseteq H_n(X_n).$$

So

$$H_n(C_\ast(X)) = H_n(X_n)/\im \partial_n = H_n(X_{n+1})$$

by exactness of the left column; but as we know this is exactly $H_n(X)$! We have proven the following result.

**Theorem 16.3.** For a CW complex $X$, there is an isomorphism

$$H_\ast(C_\ast(X)) \cong H_\ast(X)$$

natural with respect to filtration-preserving maps between CW complexes.

This has an immediate and surprisingly useful corollary.

**Corollary 16.4.** Suppose that the CW complex $X$ has only even cells – that is, $X_{2k} \hookrightarrow X_{2k+1}$ is an isomorphism for all $k$. Then

$$H_\ast(X) \cong C_\ast(X).$$

That is, $H_n(X) = 0$ for $n$ odd, is free abelian for all $n$, and the rank of $H_n(X)$ for $n$ even is the number of $n$-cells.

**Example 16.5.** Complex projective space $\mathbb{C}P^n$ has a CW structure in which

$$\Sigma_{2k} \mathbb{C}P^n = \Sigma_{2k+1} \mathbb{C}P^n = \mathbb{C}P^k.$$

The attaching $S^{2k-1} \to \mathbb{C}P^k$ sends $v \in S^{2k-1} \subseteq \mathbb{C}^n$ to the complex line through $v$. So

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & \text{for } 0 \leq k \leq 2n, k \text{ even} \\ 0 & \text{otherwise} \end{cases}.$$

Finally, notice that in our proof of Theorem 16.3 we used only properties contained in the Eilenberg-Steenrod axioms. As a result, any construction of a homology theory satisfying the Eilenberg-Steenrod axioms gives you the same values on CW complexes as singular homology.