32 Proof of the orientation theorem

We are studying the way in which local homological information gives rise to global information, especially on an \( n \)-manifold \( M \). The tool was the map

\[
j : H_n(M; R) \to \Gamma(M; o_M \otimes R)
\]

sending a class \( c \) to the section of the orientation local coefficient system given at \( x \in M \) by the restriction \( j_x(c) \in H_n(M, M - x) \). We asserted that if \( M \) is compact then \( j \) is an isomorphism and that \( H_q(M) = 0 \) for \( q > n \). The proof will be by induction.

To make the induction go, we will need a refinement of this construction. Let \( A \subseteq M \) be a compact subset. A class in \( H_n(M; M - A) \) is represented by a cycle whose boundary lies outside of \( A \). It may cover \( A \) evenly. We can give meaning to this question as follows. Let \( x \in A \). Then \( M - A \subseteq M - x \), so we have a map

\[
j_{A,x} : H_n(M, M - A) \to H_n(M, M - x)
\]

that tests whether the chain covers \( x \). As \( x \) ranges over \( A \), these maps together give us a map to the group of sections of \( o_M \) over \( A \),

\[
j_A : H_n(M, M - A) \to \Gamma(A; o_M).
\]

Because \( H_n(M, M - A) \) deals with homology classes that “stretch over \( A \),” we will write

\[
H_n(M, M - A) = H_n(M|A).
\]

**Theorem 32.1.** Let \( M \) be an \( n \)-manifold and let \( A \) be a compact subset of \( M \). Then \( H_q(M|A; R) = 0 \) for \( q > n \), and the map \( j_A : H_n(M|A; R) \to \Gamma(A; o_M \otimes R) \) is an isomorphism.

Taking \( A = M \) (assuming \( M \) compact) we find that \( H_q(M; R) = 0 \) for \( q > n \) and

\[
j_M : H_n(M; R) \xrightarrow{\cong} \Gamma(M; o_M \otimes R).
\]

But the theorem covers much more exotic situations as well; perhaps \( A \) is a Cantor set in some Euclidean space, for example.

We follow [2] in proving this, and refer you to that reference for the modifications appropriate for the more general statement when \( A \) is assumed merely closed rather than compact.

First we establish two general results.

**Proposition 32.2.** Let \( A \) and \( B \) be closed subspaces of \( M \), and suppose the result holds for \( A, B, \) and \( A \cap B \). Then it holds for \( A \cup B \).

**Proof.** The relative Mayer-Vietoris theorem and the hypothesis that \( H_{n+1}(M|A \cap B) = 0 \) gives us exactness of the top row in the ladder

\[
\begin{array}{cccc}
0 & \to & H_n(M|A \cup B) & \to & H_n(M|A) \oplus H_n(M|B) & \to & H_n(M|A \cap B) \\
& & j_{A \cap B} & & j_A \oplus j_B & & j_{A \cap B} \\
0 & \to & \Gamma(A \cup B; o_M) & \to & \Gamma(A; o_M) \oplus \Gamma(B; o_M) & \to & \Gamma(A \cap B; o_M). \\
\end{array}
\]

Exactness of the bottom row is clear: A section over \( A \cup B \) is precisely a section over \( A \) and a section over \( B \) that agree on the intersection. So the five-lemma shows that \( j_{A \cap B} \) is an isomorphism. Looking further back in the Mayer-Vietoris sequence gives the vanishing of \( H_q(M|A) \) for \( q > n \). \( \square \)
Proposition 32.3. Let \( A_1 \supseteq A_2 \supseteq \cdots \) be a decreasing sequence of compact subsets of \( M \), and assume that the theorem holds for each \( A_n \). Then it holds for the intersection \( A = \bigcap A_i \).

The proof of this proposition entails two lemmas, which we’ll dispose of first.

Lemma 32.4. Let \( A_1 \supseteq A_2 \supseteq \cdots \) be a decreasing sequence of compact subsets of a space \( X \), with intersection \( A \). Then
\[
\lim_{i \to \infty} H_q(X, X - A_i) \cong H_q(X, X - A).
\]

Proof. Let \( \sigma : \Delta^q \to X \) be any \( q \)-simplex in \( X - A \). The subsets \( X - A_i \) form an open cover of \( \text{im}(\sigma) \), so by compactness it lies in some single \( X - A_i \). This shows that
\[
\lim_{i \to \infty} S_q(X - A_i) \cong S_q(X - A).
\]

Thus
\[
\lim_{i \to \infty} S_q(X|A_i) \cong S_q(X|A_i)
\]

by exactness of direct limit, and the claim then follows for the same reason.

Lemma 32.5. Let \( A_1 \supseteq A_2 \supseteq \cdots \) be a decreasing sequence of compact subsets in a Hausdorff space \( X \) with intersection \( A \). For any open neighborhood \( U \) of \( A \) there exists \( i \) such that \( A_i \subseteq U \).

Proof. \( A \) is compact, being a closed subset of the compact Hausdorff space \( A_1 \). Since \( A \) is the intersection of the \( A_i \), and \( A \subseteq U \), the intersection of the decreasing sequence of compact sets \( A_i - U \) is empty. Thus by the finite intersection property one of them must be empty; but that says that \( A_i \subseteq U \).

Proof of Proposition 32.3. By Lemma 32.4, \( H_q(M|A) = 0 \) for \( q > n \). In dimension \( n \), we contemplate the commutative diagram
\[
\begin{array}{ccc}
\lim_{i} H_n(M|A_i) & \cong & H_n(M|A) \\
\downarrow \cong & & \downarrow \\
\lim_{i} \Gamma(A_i; o_M) & \cong & \Gamma(A; o_M).
\end{array}
\]

The top map an isomorphism by Lemma 32.4.

To see that the bottom map is an isomorphism, we’ll verify the two conditions for a map to be a direct limit from Lecture 23. First let \( x \) be a section of \( o_M \) over \( A \). By compactness, we may cover \( A \) by a finite set of opens over each of which \( o_M \) is trivial. The section extends over their union \( U \), by unique path lifting. By Lemma 32.5, this open set contains some \( A_i \), and we conclude that any section over \( A \) extends to some \( A_i \).

On the other hand, suppose that a section \( x \in \Gamma(A_i; o_M) \) vanishes on \( A \). Then it vanishes on some open set containing \( A \), again by unique path lifting and local triviality. Some \( A_j \) lies in that open set, again by Lemma 32.5. We may assume that \( j \geq i \), and conclude that \( x \) already vanishes on \( A_j \).
Proof of Theorem 32.7. There are five steps. In describing them, we will call a subset of \( M \) “Euclidean” if it lies inside some open set homeomorphic to \( \mathbb{R}^n \).

1. \( M = \mathbb{R}^n \), \( A \) a compact convex subset.
2. \( M = \mathbb{R}^n \), \( A \) a finite union of compact convex subsets.
3. \( M = \mathbb{R}^n \), \( A \) any compact subset.
4. \( M \) arbitrary, \( A \) a finite union of compact Euclidean subsets.
5. \( M \) arbitrary, \( A \) an arbitrary compact subset.

Notes on the proofs: (1) To be clear, “convex” implies nonempty. By translating \( A \), we may assume that \( 0 \in A \). The compact subset \( A \) lies in some disk, and by a homothety we may assume that the disk is the unit disk \( D^n \). Then we claim that the inclusion \( i : S^{n-1} \to \mathbb{R}^n - A \) is a deformation retract. A retraction is given by \( r(x) = x/||x|| \), and a homotopy from \( ir \) to the identity is given by

\[
h(x,t) = \left( t + \frac{1-t}{||x||} \right) x.
\]

It follows that \( H_q(\mathbb{R}^n, \mathbb{R}^n - A) \cong H_q(\mathbb{R}^n, \mathbb{R}^n - D^n) \) for all \( q \). This group is zero for \( q > n \). In dimension \( n \), note that restricting to the origin gives an isomorphism \( H_n(\mathbb{R}^n, \mathbb{R}^n - D^n) \to H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \) since \( \mathbb{R}^n - D \) is a deformation retract of \( \mathbb{R}^n - 0 \). The local system \( o_{\mathbb{R}^n} \) is trivial, since \( \mathbb{R}^n \) is simply connected, so restricting to the origin gives an isomorphism \( \Gamma(D^n, o_{\mathbb{R}^n}) \to H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \). This implies that \( j_{D^n} : H_n(\mathbb{R}^n, \mathbb{R}^n - D^n) \to \Gamma(D^n, o_{\mathbb{R}^n}) \) is an isomorphism. The restriction \( \Gamma(D^n, o_{\mathbb{R}^n}) \to \Gamma(A, o_{\mathbb{R}^n}) \) is also an isomorphism, since \( A \to D^n \) is a deformation retract. So by the commutative diagram

\[
\begin{array}{ccc}
H_n(\mathbb{R}^n, \mathbb{R}^n - D^n) & \cong & H_n(\mathbb{R}^n, \mathbb{R}^n - A) \\
\downarrow j_{D^n} & & \downarrow j_A \\
\Gamma(D^n, o_{\mathbb{R}^n}) & \longrightarrow & \Gamma(A, o_{\mathbb{R}^n})
\end{array}
\]

we find that \( j_A : H_q(\mathbb{R}^n, \mathbb{R}^n - A) \to \Gamma(A, o_{\mathbb{R}^n}) \) is an isomorphism.

(2) by Proposition 32.2.

(3) For each \( j \geq 1 \), let \( C_j \) be a finite subset of \( A \) such that

\[
A \subseteq \bigcup_{x \in C_j} B_{1/j}(x).
\]

Since any intersection of convex sets is either empty or convex,

\[
A_k = \bigcap_{j=1}^k \bigcup_{x \in C_j} B_{1/j}(x)
\]

is a union of finitely many convex sets, and since \( A \) is closed it is the intersection of this decreasing family. So the result follows from (1), (2), and Proposition 32.3.

(4) by (3) and (2).

(5) Cover \( A \) by finitely many open subsets that embed in Euclidean opens as open disks with compact closures. Their closures then form a finite cover by closed Euclidean disks \( D_i \) in Euclidean opens \( U_i \). For each \( i \), excise the closed subset \( M - U_i \) to see that

\[
H_q(M, M - A \cap D_i) \cong H_q(U_i, U_i - A \cap D_i) \cong H_q(\mathbb{R}^n, \mathbb{R}^n - A \cap D_i).
\]

By (4), the theorem holds for each of these. Each intersection \( (A \cap D_i) \cap (A \cap D_j) \) is again a compact Euclidean subset, so the result holds for them by excision as well. The result then follows by (1).
Bibliography


