4  Categorical language

Let $\text{Vect}_k$ be the category of vector spaces over a field $k$, and linear transformations between them. Given a vector space $V$, you can consider the dual $V^* = \text{Hom}(V,k)$. Does this give us a functor? If you have a linear transformation $f : V \rightarrow W$, you get a map $f^* : W^* \rightarrow V^*$, so this is like a functor, but the induced map goes the wrong way. This operation does preserve composition and identities, in an appropriate sense. This is an example of a contravariant functor.

I’ll leave it to you to spell out the definition, but notice that there is a universal example of a contravariant functor out of a category $C$: $C \rightarrow C^{\text{op}}$, where $C^{\text{op}}$ has the same objects as $C$, but $C^{\text{op}}(X,Y)$ is declared to be the set $C(Y,X)$. The identity morphisms remain the same. To describe the composition in $C^{\text{op}}$, I’ll write $f^{\text{op}}$ for $f \in C(Y,X)$ regarded as an element of $C^{\text{op}}(X,Y)$; then $f^{\text{op}} \circ g^{\text{op}} = (g \circ f)^{\text{op}}$.

Then a contravariant functor from $C$ to $D$ is the same thing as a (“covariant”) functor from $C^{\text{op}}$ to $D$.

Let $C$ be a category, and let $Y \in \text{ob}(C)$. We get a map $C^{\text{op}} \rightarrow \text{Set}$ that takes $X \mapsto C(X,Y)$, and takes a map $X \rightarrow W$ to the map defined by composition $C(W,Y) \rightarrow C(X,Y)$. This is called the functor represented by $Y$. It is very important to note that $C(-,Y)$ is contravariant, while, on the other hand, for any fixed $X$, $C(X,-)$ is a covariant functor (and is said to be “corepresentable” by $X$).

Example 4.1. Recall that the simplex category $\Delta$ has objects the totally ordered sets $[n] = \{0, 1, \ldots, n\}$, with order preserving maps as morphisms. The “standard simplex” gives us a functor $\Delta : \Delta \rightarrow \text{Top}$. Now fix a space $X$, and consider

$$[n] \mapsto \text{Top}(\Delta^n, X).$$

This gives us a contravariant functor $\Delta \rightarrow \text{Set}$, or a covariant functor $\Delta^{\text{op}} \rightarrow \text{Set}$. This functor carries in it all the face and degeneracy maps we discussed earlier, and their compositions. Let us make a definition.
Definition 4.2. Let $C$ be any category. A simplicial object in $C$ is a functor $K : \Delta^{op} \to C$. Simplicial objects in $C$ form a category with natural transformations as morphisms. Similarly, semi-simplicial object in $C$ is a functor $\Delta^{op}_{\text{inj}} \to C$.

So the singular functor $\text{Sin}_*$ gives a functor from spaces to simplicial sets (and so, by restriction, to semi-simplicial sets).

I want to interject one more bit of categorical language that will often be useful to us.

Definition 4.3. A morphism $f : X \to Y$ in a category $C$ is a split epimorphism ("split epi" for short) if there exists $g : Y \to X$ (called a section or a splitting) such that the composite $Y \xrightarrow{g} X \xrightarrow{f} Y$ is the identity.

Example 4.4. In the category of sets, a map $f : X \to Y$ is a split epimorphism exactly when, for every element of $Y$ there exists some element of $X$ whose image in $Y$ is the original element. So $f$ is surjective. Is every surjective map a split epimorphism? This is equivalent to the axiom of choice! because a section of $f$ is precisely a choice of $x \in f^{-1}(y)$ for every $y \in Y$.

Every categorical definition is accompanied by a “dual” definition.

Definition 4.5. A map $g : Y \to X$ is a split monomorphism ("split mono" for short) if there is $f : X \to Y$ such that $f \circ g = 1_Y$.

Example 4.6. Again let $C = \text{Set}$. Any split monomorphism is an injection: If $y, y' \in Y$, and $g(y) = g(y')$, we want to show that $y = y'$. Apply $f$, to get $y = f(g(y)) = f(g(y')) = y'$. But the injection $\emptyset \to Y$ is a split monomorphism only if $Y = \emptyset$. So there’s an asymmetry in the category of sets.

Lemma 4.7. A map is an isomorphism if and only if it is both a split epimorphism and a split monomorphism.

Proof. Easy! □

The importance of these definitions is this: Functors will not in general respect “monomorphisms” or “epimorphisms,” but:

Lemma 4.8. Any functor sends split epis to split epis and split monos to split monos.

Proof. Apply $F$ to the diagram establishing $f$ as a split epi or mono. □

Example 4.9. Suppose $C = \text{Ab}$, and you have a split epi $f : A \to B$. Let $g : B \to A$ be a section. We also have the inclusion $i : \ker f \to A$, and hence a map

$[g \ i] : B \oplus \ker f \to A$.

I leave it to you to check that this map is an isomorphism, and to formulate a dual statement.