6 Homotopy invariance of homology

We now know that the homology of a star-shaped region is trivial: in such a space, every cycle with augmentation 0 is a boundary. We will use that fact, which is a special case of homotopy invariance of homology, to prove the general result, which we state in somewhat stronger form:

Theorem 6.1. A homotopy \( h : f_0 \simeq f_1 : X \to Y \) determines a natural chain homotopy \( f_0 * \simeq f_1 * : S_*(X) \to S_*(Y) \).

The proof uses naturality (a lot). For a start, notice that if \( k : g_0 \simeq g_1 : C_* \to D_* \) is a chain homotopy, and \( j : D_* \to E_* \) is another chain map, then the composites \( j \circ k_n : C_n \to E_{n+1} \) give a chain homotopy \( j \circ g_0 \simeq j \circ g_1 \). So if we can produce a chain homotopy between the chain maps induced by the two inclusions \( i_0, i_1 : X \to X \times I \), we can get a chain homotopy \( k \) between \( f_0 = h_s \circ i_0 \) and \( f_1 = h_s \circ i_1 \) in the form \( h_s \circ k \).

So now we want to produce a natural chain homotopy, with components \( k_n : S_n(X) \to S_{n+1}(X \times I) \). The unit interval hosts a natural 1-simplex given by an identification \( \Delta^1 \to I \), and we should imagine \( k \) as being given by “multiplying” by that 1-chain. This “multiplication” is a special case of a chain map

\[
\times : S_*(X) \times S_*(Y) \to S_*(X \times Y),
\]
defined for any two spaces \( X \) and \( Y \), with lots of good properties. It will ultimately be used to compute the homology of a product of two spaces in terms of the homology groups of the factors.

Here’s the general result.

Theorem 6.2. There exists a map \( \times : S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y) \), the cross product, that is:

- **Natural**, in the sense that if \( f : X \to X' \) and \( g : Y \to Y' \), and \( a \in S_p(X) \) and \( b \in S_p(Y) \) so that \( a \times b \in S_{p+q}(X \times Y) \), then \( f_*(a) \times g_*(b) = (f \times g)_*(a \times b) \).

- **Bilinear**, in the sense that \( (a + a') \times b = (a \times b) + (a' \times b) \), and \( a \times (b + b') = a \times b + a \times b' \).

- **The Leibniz rule is satisfied**, i.e., \( d(a \times b) = (da) \times b + (-1)^p a \times db \).

- **Normalized, in the following sense.** Let \( x \in X \) and \( y \in Y \). Write \( j_x : Y \to X \times Y \) for \( y \mapsto (x, y) \), and write \( i_y : X \to X \times Y \) for \( x \mapsto (x, y) \). If \( b \in S_q(Y) \), then \( c_\times^0 \times b = (j_x)_* b \in S_q(X \times Y) \), and if \( a \in S_p(X) \), then \( a \times c_y^0 = (i_y)_* a \in S_p(X \times Y) \).

The Leibniz rule contains the first occurrence of the “topologist’s sign rule”; we’ll see these signs appearing often. Watch for when it appears in our proof.

Proof. We’re going to use induction on \( p + q \); the normalization axiom gives us the cases \( p + q = 0, 1 \). Let’s assume that we’ve constructed the cross-product in total dimension \( p + q - 1 \). We want to define \( \sigma \times \tau \) for \( \sigma \in S_p(X) \) and \( \tau \in S_q(Y) \).

Note that there’s a universal example of a \( p \)-simplex, namely the identity map \( \iota_p : \Delta^p \to \Delta^p \). It’s universal in the sense any \( p \)-simplex \( \sigma : \Delta^p \to X \) can be written as \( \sigma_* \iota_p \) where \( \sigma_* : \text{Sim}_p(\Delta^p) \to \text{Sim}_p(X) \) is the map induced by \( \sigma \). To define \( \sigma \times \tau \) in general, then, it suffices to define \( \iota_p \times \iota_q \in S_{p+q}(\Delta^p \times \Delta^q) \); we can (and must) then take \( \sigma \times \tau = (\sigma \times \tau)_* (\iota_p \times \iota_q) \).

Our long list of axioms is useful in the induction. For one thing, if \( p = 0 \) or \( q = 0 \), normalization provides us with a choice. So now assume that both \( p \) and \( q \) are positive. We want the cross-product to satisfy the Leibnitz rule:

\[
d(\iota_p \times \iota_q) = (d\iota_p) \times \iota_q + (-1)^p \iota_p \times d\iota_q \in S_{p+q-1}(\Delta^p \times \Delta^q)
\]

Since \( d^2 = 0 \), a necessary condition for \( \iota_p \times \iota_q \) to exist is that \( d((d\iota_p) \times \iota_q + (-1)^p \iota_p \times d\iota_q) = 0 \). Let’s compute what this is, using the Leibnitz rule in dimension \( p + q - 1 \) where we have it by the inductive assumption:

\[
d((d\iota_p) \times \iota_q + (-1)^p \iota_p \times (d\iota_q)) = (d^2 \iota_p) \times \iota_q + (-1)^{p-1}(d\iota_p) \times (d\iota_q) + (d\iota_p) \times (d\iota_q) + (-1)^q \iota_p \times (d^2 \iota_q) = 0
\]
because \( d^2 = 0 \). Note that this calculation would not have worked without the sign!
The subspace \( \Delta^p \times \Delta^q \subseteq \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} \) is convex and nonempty, and hence star-shaped. Therefore we know that \( H_{p+q-1}(\Delta^p \times \Delta^q) = 0 \) (remember, \( p + q > 1 \)), which means that every cycle is a boundary. In other words, our necessary condition is also sufficient! So, choose any element with the right boundary and declare it to be \( t_p \times t_q \).

The induction is now complete provided we can check that this choice satisfies naturality, bilinearity, and the Leibniz rule. I leave this as a relaxing exercise for the listener.

The essential point here is that the space supporting the universal pair of simplices – \( \Delta^p \times \Delta^q \) – has trivial homology. Naturality transports the result of that fact to the general situation.

The cross-product that this procedure constructs is not unique; it depends on a choice of the chain \( t_p \times t_q \) for each pair \( p, q \) with \( p + q > 1 \). The cone construction in the proof that star-shaped regions have vanishing homology provides us with a specific choice; but it turns out that any two choices are equivalent up to natural chain homotopy.

We return to homotopy invariance. To define our chain homotopy \( h_X : S_n(X) \to S_{n+1}(X \times I) \), pick any 1-simplex \( \iota : \Delta^1 \to I \) such that \( d_0 \iota = c_1^0 \) and \( d_1 \iota = c_0^0 \), and define

\[
h_X \sigma = (-1)^n \sigma \times \iota.
\]

Let’s compute:

\[
dh_X \sigma = (-1)^n d(\sigma \times \iota) = (-1)^n (d\sigma) \times \iota + \sigma \times (d\iota)
\]

But \( d\iota = c_1^0 - c_0^0 \in S_0(I) \), which means that we can continue (remembering that \( |\partial \sigma| = n - 1 \)):

\[
= -h_X d\sigma + (\sigma \times c_1^0 - \sigma \times c_0^0) = -h_X d\sigma + (\iota_1 \sigma - \iota_0 \sigma),
\]

using the normalization axiom of the cross-product. This is the result.