HOMEWORK 6

DUE: MONDAY, 3/20/06

1. (Hatcher) (a) Show that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$.
   (b) Show there is a map $\mathbb{RP}^\infty \to \mathbb{CP}^\infty$ which induces the trivial map on $H_*(-)$ but a nontrivial map on $H^*(-)$. How is this consistent with the universal coefficient theorem?

2. (Hatcher) Given abelian groups $G$ and $H$ and CW complexes $K(G, n)$ and $K(H, n)$, show that the map $[K(G, n), K(H, n)]_* \to \text{Hom}(G, H)$ sending a homotopy class $[f]$ to the induced homomorphism $f_* : \pi_n K(G, n) \to \pi_n K(H, n)$ is a bijection.

3. (This may be useful for the next problem) Let $f : X \to Y$ be a pointed map. Show that the cofiber of
   $$f \wedge 1 : X \wedge Z \to Y \wedge Z$$
   is given by $C(f) \wedge Z$.

4. Let $n$ be greater than 1. Assuming that there is a natural isomorphism
   $$\tilde{H}_{k+n}(X \wedge M(\pi, k)) \cong \tilde{H}_n(X, \pi)$$
   show that the universal coefficient theorem follows from the long exact sequence of the cofiber sequence
   $$\bigvee_i S^n_i \to \bigvee_i S^n_i \to M(\pi, n).$$

The Snake lemma may be useful in the following two problems:

5. A directed system $\{A_i\}$ of abelian groups is a sequence of homomorphisms
   $$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots$$
   A map of directed systems
   $$\{A_i\} \to \{B_i\}$$
   is a sequence of homomorphisms $A_i \to B_i$ making the diagrams
   $$\begin{array}{ccc}
   A_i & \to & A_{i+1} \\
   \downarrow & & \downarrow \\
   B_i & \to & B_{i+1}
   \end{array}$$
   commute. A short exact sequence of directed systems
   $$0 \to \{A_i\} \to \{B_i\} \to \{C_i\} \to 0$$
is a short exact sequence at every level
\[ 0 \to A_i \to B_i \to C_i \to 0 \]

(a) Show that \( \lim A_i \) is given by the kernel of the map
\[ \phi : \bigoplus A_i \to \bigoplus A_i \]
where \( \phi(\sum a_i) = \sum f_i(a_i) + a_i \).
(b) Show that \( \lim \) is an exact functor from the category of directed systems of abelian groups to the category of abelian groups. That is to say, the direct limit of a short exact sequence of directed systems is a short exact sequence.

6. In a manner precisely analogous to the previous problem, you can consider the category of inverse systems of abelian groups
\[ A_1 \xleftarrow{f_1} A_2 \xleftarrow{f_2} A_3 \xleftarrow{f_3} \ldots \]

(a) Show that a short exact sequence of inverse systems
\[ 0 \to \{A_i\} \to \{B_i\} \to \{C_i\} \to 0 \]
gives rise to an exact sequence
\[ 0 \to \lim A_i \to \lim B_i \to \lim C_i \to \lim^1 A_i \to \lim^1 B_i \to \lim^1 C_i \to 0 \]
(b) Show that for a prime \( p \), the sequence
\[ \mathbb{Z} \xleftarrow{p} \mathbb{Z} \xleftarrow{p} \mathbb{Z} \xleftarrow{p} \ldots \]
has
\[ \lim = 0 \]
\[ \lim^1 = \mathbb{Z}_p / \mathbb{Z} \]
Here, \( \mathbb{Z}_p = \lim \mathbb{Z} / p^i \) are the \( p \)-adic integers.