HOMEWORK 7

DUE DATE: WEDNESDAY, APRIL 5

1. Argue that there exists a map $\alpha : S^2 \to S^2 \vee S^1$ so that the inclusion
   $S^1 \hookrightarrow (S^2 \vee S^1) \cup_\alpha D^3$
   induces an isomorphism on $\pi_1$ and $\tilde{H}_*$, but is not a homotopy equivalence.

2. (Hatcher) Consider the equivalence relation $\sim_w$ generated by weak homotopy equivalence: $X \sim_w Y$ if there are spaces $X = X_1, X_2, \ldots, X_n = Y$ with weak homotopy equivalences $X_i \to X_{i+1}$ or $X_i \leftarrow X_{i+1}$ for each $i$. Show that $X \sim_w Y$ iff $X$ and $Y$ have a common CW-approximation.

3. Show that $p : E \to B$ is a principle $G$-bundle, and $f : X \to B$ is a map, then the pullback $f^*E = E \times_B X \to X$ is a principle $G$-bundle. Show that if $g : Y \to X$ is another map, then there is an isomorphism of $G$-bundles
   $g^*f^*E \cong (f \circ g)^*E$.

4. Suppose that $X$ is a pointed, connected CW complex. A trivialization of a principle $G$-bundle
   $p : E \to X$
   is a section $s : X \to E$, satisfying $ps = Id_X$. An isomorphism of trivialized bundles is a bundle isomorphism which preserves the trivialization.

   (a) Argue that a trivialization is the same thing as an isomorphism with the trivial $G$-bundle.
   (b) Let $EG \to BG$ be the universal $G$-bundle. Argue that there is an isomorphism
       $[X, EG]_* \cong \{\text{isomorphism classes of trivialized } G\text{-bundles}\}$.
   (c) Show that any two trivialized bundles are isomorphic. Conclude that $EG$ is contractible, and that $\Omega BG$ is weakly equivalent to $G$.

WARNING: This problem turns out to be subtler than I initially imagined. You may for simplicity assume that $G$ is discrete!