All base spaces are assumed to be paracompact.

1. (a) Show that there is a homeomorphism
   \[(X \times Y)^{V \oplus W} \approx X^V \wedge Y^W.\]

(b) Deduce that there is a homeomorphism
   \[X^{V \oplus \mathbb{R}^k} \approx \Sigma^k X^V\]
   where \(\mathbb{R}^k\) is the trivial bundle over \(X\).

2. Show that if \(V\) is a vector bundle with a non-vanishing section, then the Euler class \(e(V)\) must vanish. (Note: if \(X\) were a manifold, then this would be what you would expect from the geometric description I gave you in class.)

Gysin maps

The next two problems investigate a map which goes the “wrong way” in cohomology called the Gysin map. From now on we always work with homology with mod 2 coefficients to avoid having to discuss orientations, and manifolds are assumed to be smooth, connected, closed, and compact.

Let \(i : N \hookrightarrow M\) be the inclusion of a submanifold of a manifold \(M\), with \(\dim N = n\) and \(\dim M = m\). Give the tangent bundle \(TM\) a metric, and define \(\nu = TN^\perp\) to be the normal bundle of \(N\) in \(TM\). The “tubular neighborhood theorem” of differential topology asserts that there is a tubular neighborhood \(\text{Tube}(N)\) of \(N\) in \(M\) whose closure \(\overline{\text{Tube}(N)}\) is diffeomorphic to the disk bundle \(D(\nu)\). Let

\[P : M \to \overline{\text{Tube}(N)}/\partial \text{Tube}(N) \approx N^\nu\]

be the map which sends all points outside of \(\text{Tube}(N)\) to the basepoint. This map is called the Pontryagin-Thom collapse map. It induces, via the Thom isomorphism, a map going in the wrong way called a Gysin map:

\[i_! : H^*(N) \cong \tilde{H}^{*+m-n}(N^\nu) \overset{P^*}{\longrightarrow} H^{*+m-n}(M).\]

In particular, we get a (mod 2) cohomology class \([N]\) whose dimension is the codimension of \(N\) in \(M\):

\[[N] := i_!(1) \in H^{m-n}(M).\]

3. Verify that for the inclusion of a point \(* \hookrightarrow M\), the class \([*] \in H^m(M)\) is dual to the fundamental class \([M] \in H_m(M)\).

4. A pair of submanifolds \(N_1\) and \(N_2\) of dimensions \(n_1\) and \(n_2\), respectively, are said to be transverse in \(M\) if for each point \(x \in N_1 \cap N_2\), the tangent space \(TM_x\)
is spanned by the subspaces $(TN_1)_x$ and $(TN_2)_x$. The implicit function theorem then may be used to show that $N_1 \cap N_2$ is a submanifold of dimension $n_1 + n_2 - m$, with tangent bundle $TN_1 \cap TN_2 \hookrightarrow TM$.

Verify the formula

$$[N_1] \cup [N_2] = [N_1 \cap N_2] \in H^{2m-n_1-n_2}(M).$$

In other words, for geometric cocycles in general position, the cup product is given by intersection.