LECTURE 4: SIMPLE COMPUTATIONS, THE ACTION OF THE FUNDAMENTAL GROUPOID

1. Simple computations

By using smooth or simplicial approximation, we can prove:

**Proposition 1.1.** For $k < n$, we have $\pi_k(S^n) = 0$.

**Proposition 1.2.** Suppose that $p_* : X \to Y$ is a covering space. Then the induced map

$$p_* : \pi_k(X) \to \pi_k(Y)$$

is a monomorphism for $k = 1$ and an isomorphism for $k > 1$.

Using the covering $\mathbb{R} \to S^1$, we have

$$\pi_k(S^1) = \begin{cases} \mathbb{Z}, & k = 1 \\ 0, & k \neq 1 \end{cases}$$

In fact, any space with a contractible universal cover has trivial higher homotopy groups. For instance, the torus $T^2$:

$$\pi_k(T^2) = \begin{cases} \mathbb{Z} \times \mathbb{Z}, & k = 1 \\ 0, & k \neq 1 \end{cases}$$

This result can be obtained in a different way using $T^2 = S^1 \times S^1$ together with the following lemma.

**Lemma 1.3.** There is an isomorphism $\pi_k(X \times Y) \cong \pi_k(X) \times \pi_k(Y)$.

In general we shall (eventually) see that $\pi_n(S^n) = \mathbb{Z}$. In general, the groups $\pi_k(S^n)$ for $k > n$ are a mess. They are known through a finite range, and are very difficult to compute.

Given a sequence of sets

$$T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} T_2 \xrightarrow{f_2} \cdots$$

we can form the colimit or direct limit:

$$\varinjlim T_i = \left( \prod_i T_i \right) / / (t \sim f_i(t)).$$

- If each of the $f_i$’s are inclusions, then $\varinjlim T_i$ is the union.
- If the $f_i$’s are continuous maps between topological spaces, then there is a natural quotient topology that can be placed on $\varinjlim T_i$.
- If each of the $f_i$’s is a closed inclusion of compactly generated spaces, then $\varinjlim T_i$ is compactly generated, and has the topology of the union.
- If each of the $f_i$’s is a homomorphism between groups, then $\varinjlim T_i$ is a group.

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Lemma 1.4. Let

\[ X_0 \to X_1 \to X_2 \to \cdots \]

be a sequence of closed inclusions. Let \( K \) be a compact space. Then there is a bijection

\[ \lim \operatorname{Map}(K, X_i) \cong \operatorname{Map}(K, \lim X_i). \]

Corollary 1.5. There is an isomorphism of groups

\[ \lim \pi_k(X_i) \cong \pi_k(\lim X_i). \]

Let \( S^\infty \) be the colimit \( \lim \n_{\infty} S^n \). Proposition 1.1 and Corollary 1.5 combine to yield

\[ \pi_k(S^\infty) = 0. \]

Since \( S^\infty \to \mathbb{R}P^\infty \) is a two-fold cover, we have

\[ \pi_k(\mathbb{R}P^\infty) = \begin{cases} \mathbb{Z}/2, & k = 1 \\ 0, & k \neq 1 \end{cases} \]

In general, if \( \pi \) is a group (abelian if \( n > 1 \)), then if \( X \) is a space satisfying

\[ \pi_k(X) = \begin{cases} \pi, & k = 1 \\ 0, & k \neq 1 \end{cases} \]

it is called an Eilenberg-MacLane space or a \( K(\pi, n) \). Usually we assume also that \( X \) is a CW complex. We shall see that such spaces are unique up to homotopy.

2. The action of the fundamental groupoid

A groupoid is a category for which every morphism is an isomorphism. A group is a groupoid with one object; given a group \( G \), we have a category with one object \( * \), and morphisms \( \operatorname{Map}(*,*) := G \). Group multiplication gives composition, and the existence of inverses implies that this category is in fact a groupoid.

Let \( X \) be an unpointed space. Define its fundamental groupoid \( \pi_{\text{oid}}(X) \) to be the category whose objects are the points of \( X \), and whose morphisms \( x \to y \) are given by paths from \( x \) to \( y \) modulo homotopy relative to the endpoints. This is easily seen to be a groupoid.

We showed that given a path \( \gamma : x \to y \), we have an induced homomorphism

\[ \gamma_* : \pi_k(X, x) \to \pi_k(X, y). \]

The homomorphism \( \gamma_* \) is easily seen to depend only on the class it represents in the fundamental groupoid.

Proposition 2.1. This action yields a functor

\[ \pi_k(X, -) : \pi_{\text{oid}}(X) \to \text{Groups} \]

\[ x \mapsto \pi_k(X, x) \]

\[ ([\gamma] : x \to y) \mapsto (\gamma_* : \pi_k(X, x) \to \pi_k(X, y)). \]

In particular, there is an action of \( \pi_1(X, x) \) on \( \pi_k(X, x) \).