LECTURE 21: SPECTRAL SEQUENCES

In homological algebra, a short exact sequence of chain complexes gives rise to a long exact sequence of homology groups. You could think of a short exact sequence as a 2-stage filtration of a chain complex. A more arbitrary filtered chain complex gives rise to a generalized version of a long exact sequence, called a spectral sequence.

**Definition 0.1.** A *(increasing)* filtered graded abelian group is a pair \((A_*, \{F_sA_*\}_s)\) where \(A_*\) is a graded abelian group, and a sequence of sub-groups

\[
F_0A_n \subseteq F_1A_n \subseteq F_2A_n \subseteq \cdots
\]

such that \(A_n = \lim_{s \to \infty} F_sA_n\). By convention, \(F_sA_n := 0\) for \(s < 0\).

Given a filtered graded abelian group \(\{F_sA_*\}\), the successive quotients give rise to a bigraded group \(Gr_*A_*\) called the associated graded group:

\[
Gr_*A_n := F_sA_n/F_{s-1}A_n.
\]

**Definition 0.2.** A filtered chain complex \((C_*, \{F_sC_*\}, d)\) is a chain complex \(C_*\) and a sequence of sub-chain complexes \(F_sC_*\), such that \((C_*, \{F_sC_*\})\) is a filtered graded abelian group.

Since the differential preserves the filtration, the associated graded of a filtered chain complex \(\{F_sC_*\}\) inherits the structure of a chain complex

\[
d : Gr_*C_n \to Gr_*C_{n-1}.
\]

The homology of \(C_*\) also inherits a filtration:

\[
F_*H_n(C) := \text{im}(H_n(F_sC) \to H_n(C)).
\]

The question is the following: what is the relationship between

1. the homology of the associated graded \(H_*(Gr_*C_*)\), and
2. the associated graded of the homology \(Gr_*H_*(C_*)\)?

The answer is that there is a spectral sequence

\[
E^1_{s,t} = H_{s+t}(Gr_*C_*) \Rightarrow H_{s+t}(C_*)
\]

What does this terminology mean?

**Definition 0.3.** A spectral sequence *(of homological type)* is a sequence of differential bigraded abelian groups

\[
\{E^r_{s,t} \}_{s,t \in \mathbb{Z}, r}.
\]

The index \(r\) begins somewhere, typically \(r \geq 1\) or \(r \geq 2\). The differential \(d_r\) is a homomorphism

\[
d_r : E^r_{s,t} \to E^r_{s-r,t+r-1}
\]

*Date: 4/4/06.*
which satisfies \( d_r^2 = 0 \). For a fixed \( r \), the bigraded abelian group \( E_{s, t}^r \) is called the \( r \)th page of the spectral sequence. Each page is required to be the homology of the previous page:

\[
E_{s, t}^{r+1} = H_s(E_{s, t}^r, d_r).
\]

A spectral sequence is called first quadrant of there exists an \( r \) so that \( E_{s, t}^r \) is concentrated in the first quadrant, that is:

\[
E_{s, t}^r = 0 \quad \text{if } s < 0 \text{ or } t < 0.
\]

All of the spectral sequences we will be dealing with will turn out to be first quadrant, so let’s make this assumption for now on. This condition implies that for fixed \( s, t \), the maps

\[
d_r : E_{s, t}^r \to E_{s-r, t+r-1}^r
\]

\[
d_r : E_{s+r, t-r+1}^r \to E_{s, t}^r
\]

are zero for \( r >> 0 \). This implies that for fixed \( s, t \) and large \( r \), we have

\[
E_{s, t}^r = E_{s, t}^{r+1}.
\]

These stable values are denoted \( E_{s, t}^\infty \).

Let \( A_* \) be a graded abelian group. We use the notation

\[
E_{s, t}^r \Rightarrow A_{s+t}
\]

and say that the spectral sequence converges to \( A_* \) provided there exists a filtration \( \{F_s A_*\} \) on \( A_* \) and an isomorphism

\[
E_{s, t}^\infty \cong Gr_s A_{s+t}.
\]