LECTURE 31: COMPLETION OF A DEFERRED PROOF, 
WHITNEY SUM, AND CHERN CLASSES

1. A DEFERRED PROOF

From the last lecture, I constructed for a sub-Lie group $H$ of a Lie group $G$ a map
\[ \phi: EG/H \to BH, \]
and owed you a proof of

**Proposition 1.1.** The map $\phi$ is an equivalence, and the map induced by the inclusion
\[ i: H \hookrightarrow G \]
is the quotient map
\[ i_*: BH \simeq EG/H \to EG/G = BG. \]

We first state some easy lemmas.

**Lemma 1.2.** The induction functor
\[ \text{Ind}_H^G = G \times_H (-): H\text{-spaces} \to G\text{-spaces} \]
is left adjoint to the restriction functor
\[ \text{Res}_H^G: G\text{-spaces} \to H\text{-spaces} \]
which regards a $G$-space $X$ as an $H$-space. In particular, there is a natural isomorphism
\[ \text{Map}_G(G \times_H X, Y) \cong \text{Map}_H(X, Y) \]
for an $H$-space $X$ and a $G$-space $Y$.

Given $G$-bundles $E \to B$ and $E' \to B'$, a $G$-equivariant map
\[ f: E \to E' \]
gives rise to a map of bundles
\[ E \xrightarrow{f} E' \]
\[ B \xrightarrow{i/G} B' \]

**Lemma 1.3.** There is an equivalence of bundles
\[ E \cong (f/G)^* E'. \]
Sketch proof of Proposition 1.1. We need to construct a map in the opposite direction. The $G$-bundle $G \times_H EH \to BH$ is classified by a map

$$G \times_H EH \xrightarrow{f} EG$$

$$BH \xrightarrow{} BG.$$

Let

$$\tilde{f} : EH \to EG$$

be the $H$-equivariant map adjoint to the map $f$. Let $\psi$ be the induced map of $H$-orbits:

$$\psi = \tilde{f}/H : BH = EH/H \to EG/H.$$

The composite $\phi \circ \psi$ is seen to be an equivalence because it is covered by the $H$-equivariant composite

$$EH \xrightarrow{\tilde{f}} EG \to EH$$

and thus classifies the universal bundle over $BH$.

The composite $\psi \circ \phi$ is covered by the $H$-equivariant composite

$$\tilde{h} : EG \to EH \xrightarrow{\tilde{f}} EG$$

whose adjoint $h$ gives a map of $G$-bundles

$$G \times_H EG \xrightarrow{h} EG$$

$$EG/H \xrightarrow{\tilde{h}} BG$$

The bundle $G \times_H EG \to EG/H$ is easily seen to be classified by the quotient map $EG/H \to EG/G = BG$. Thus we can conclude that $h$ is $G$-equivariantly homotopic to the map

$$G \times_H EG \to EG$$

which sends $[g, e]$ to $ge$. Thus the adjoint $\tilde{h}$ is $H$-equivariantly homotopic to the identity map $EG \to EG$. Taking $H$-orbits, we see that $\psi \circ \phi$ is homotopic to the identity.

\[\square\]

2. Whitney sum

Let $V$ be an $n$-dimensional complex vector bundle over $X$ and $W$ be an $m$-dimensional complex vector bundle over $Y$.

Definition 2.1. The external direct sum $V \boxplus W$ is the product bundle

$$V \boxplus W = V \times W \to X \times Y$$

where the vector space structure on the fibers is given by the direct sum.

Now assume $X = Y$. 2
**Definition 2.2.** The Whitney sum $V \oplus W$ is the $n+m$-dimensional complex vector bundle given by the pullback $\Delta^* V \boxplus W$, where

$$\Delta : X \rightarrow X \times X$$

is the diagonal.

Let $V_{\text{univ}}^n$ be the universal $n$-dimensional complex vector bundle over $BU(n)$. Let

$$f_{n,m} : BU(n) \times BU(m) \rightarrow BU(n+m)$$

be the classifying map of $V_{\text{univ}}^n \boxplus V_{\text{univ}}^m$. Then if

$$f_V : X \rightarrow BU(n)$$
$$f_W : X \rightarrow BU(m)$$

classify $V$ and $W$, respectively, the composite

$$X \xrightarrow{f_V \times f_W} BU(n) \times BU(m) \xrightarrow{f_{n,m}} BU(n+m)$$

classifies $V \oplus W$.

3. **Chern classes**

Our computation of $H^*(BU(n))$ allows for the definition of characteristic classes for complex vector bundles.

**Definition 3.1.** Let $V \rightarrow X$ be a complex $n$-dimensional vector bundle, with classifying map

$$f_V : X \rightarrow BU(n).$$

We define the $i$th Chern class $c_i(V) \in H^{2i}(X; \mathbb{Z})$ to be the induced class $f_V^*(c_i)$ for $1 \leq i \leq n$. We use the following conventions:

$$c_0(V) := 1$$
$$c_i(V) := 0 \quad \text{for} \ i > n.$$  

These classes are natural: for a map $f : Y \rightarrow X$ we have

$$c_i(f^*V) = f^*c_i(V).$$