LECTURE 32: PROPERTIES OF CHERN CLASSES, THE
SPLITTING PRINCIPLE

1. Axioms for Chern classes

The Chern classes $c_i(V)$ of vector bundles satisfy the following axioms.

**Naturality:** For $V$ a complex vector bundle over $Y$, and a map $f : X \to Y$, we have

$$c_i(f^* V) = f^* c_i(V).$$

**Stability:** We have $c_i(V \oplus \mathbb{C}) = c_i(V)$, where $\mathbb{C}$ is the trivial bundle.

**Dimension:** If $\dim \mathbb{C} V = n$, then $c_i(V) = 0$ for $i > n$.

**Sum formula:** For complex vector bundles $V$ and $W$ over $X$, we have

$$c_i(V \oplus W) = \sum_{i_1 + i_2 = i} c_{i_1}(V) \cup c_{i_2}(W).$$

**Normalization:** For the universal line bundle $L_{univ} \to \mathbb{C}P^\infty$, the first Chern class $c_1(L_{univ})$ agrees with the generator $c_1 \in H^2(\mathbb{C}P^\infty)$.

The naturality, dimension, and normalization axioms are immediate consequences of our definition of the Chern classes. Stability follows from our inductive calculation of the cohomology of $BU(n)$: analysis of the edge homomorphism tells us that the map

$$\mathbb{Z}[c_1, \ldots, c_n] = H^*(BU(n)) \to H^*(BU(n-1)) = \mathbb{Z}[c_1, \ldots, c_{n-1}]$$

sends $c_i$ to $c_i$ for $i < n$ and that $c_n$ is mapped to zero.

We are left with the sum formula, which is the most important fact about Chern classes. We will first prove the sum formula up to a constant using the splitting principle. We will then determine the constant using the “Euler class”.

2. The splitting principle

The splitting principle essentially says

Any universal formula involving Chern classes need only be checked on sums of line bundles.

The actual statement is as follows.

**Theorem 2.1** (The splitting principle). Let $V$ be an $n$-dimensional complex vector bundle over a space $X$. There exists a space $\tilde{X}$ together with a map $f : \tilde{X} \to X$ so that

1. The pullback of $V$ is given by

$$f^*(V) \cong L_1 \oplus \cdots \oplus L_n$$

where the $L_i$ are complex line bundles over $\tilde{X}$.

2. The map $f^* : H^*(X) \to H^*(\tilde{X})$ is injective.

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Proof. We prove the theorem by splitting off one line at a time. Let
\[ \mathbb{C}P^n \to P(V) \to X \]
be the projective space bundle given by the space
\[ P(V) = \{(x, L) : x \in X, \text{ and } L \text{ is a line in the fiber } V_x \}. \]
There is a canonical line subbundle in \( g^*V \) whose fiber over a pair \((x, L)\) is the line \( L \). Endowing \( g^*V \) with a Hermitian structure, and letting \( L^\perp \) be the orthogonal complement of \( L \) in \( g^*V \), we have a decomposition
\[ g^*V \cong L \oplus L^\perp. \]
The map \( f \) is seen to be an injection using the edge homomorphism of the cohomological Serre spectral sequence for the fibration \( f \). There is room for non-trivial differentials if \( X \) has odd-dimensional cohomology, but these are shown to be zero using naturality and the universal example where \( X = BU(n) \) and \( V = V_{\text{univ}} \). \( \square \)