18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.
Steenrod Operations (Lecture 2)

The objective of today’s lecture is to introduce the Steenrod operations and establish some of their basic properties. We will work over the finite field $\mathbb{F}_2 \simeq \mathbb{Z}/2\mathbb{Z}$ with two elements.

To this end, we will study the homotopy theory of cochain complexes

$$\ldots \rightarrow V^{n-1} \xrightarrow{d_{n-1}} V^n \xrightarrow{d_n} V^{n+1} \rightarrow \ldots$$

in the category of $\mathbb{F}_2$-vector spaces. We will refer to these objects simply as complexes. To each complex $V$ we can associate cohomology groups

$$H^n V = \ker(d_n)/\text{Im}(d_{n-1}).$$

**Remark 1.** It is possible to take a more sophisticated point of view: we can identify cochain complexes $V$ over the field $\mathbb{F}_2$ with module spectra over $\mathbb{F}_2$. The cohomology groups $H^n(V)$ should then be viewed as the homotopy groups $\pi_n$ of the corresponding spectra.

Given a pair of $\mathbb{F}_2$-module spectra $V$ and $W$, we can form their tensor product $V \otimes W$. This is given by the usual tensor product of complexes of vector spaces:

$$(V \otimes W)^n = \oplus_{n=n'+n''} V^{n'} \otimes W^{n''},$$

with the usual differential (note that, since we are working over the field $\mathbb{F}_2$, we do not even have to worry about signs). In particular, we can form the tensor powers

$$V^\otimes n = V \otimes V \otimes \ldots \otimes V$$

of a fixed $\mathbb{F}_2$-module spectrum. The tensor power $V^\otimes n$ inherits a natural action of the symmetric group $\Sigma_n$, by permuting the tensor factors.

One of the most important examples of an $\mathbb{F}_2$-module spectrum is the cochain complex

$$C^*(X; \mathbb{F}_2)$$

of a topological space $X$. The cohomology groups of this $\mathbb{F}_2$-module spectrum are simply the cohomology groups of $X$. The cohomology $H^*(X; \mathbb{F}_2)$ has the structure of a graded commutative ring. The multiplication on $H^*(X; \mathbb{F}_2)$ arises from a multiplication which exists on the cochain complex $C^*(X; \mathbb{F}_2)$. Namely, we can consider the composition

$$C^*(X; \mathbb{F}_2) \otimes C^*(X; \mathbb{F}_2) \rightarrow C^*(X \times X; \mathbb{F}_2) \rightarrow C^*(X; \mathbb{F}_2).$$

Here the first map is the classical Alexander-Whitney morphism, and the second is given by pullback along the diagonal inclusion $X \rightarrow X \times X$. The Alexander-Whitney map is *not* compatible with the action of the symmetric group $\Sigma_2$ on the two sides. Consequently, the resulting multiplication

$$m : C^*(X; \mathbb{F}_2) \otimes C^*(X; \mathbb{F}_2) \rightarrow C^*(X; \mathbb{F}_2)$$
is not commutative until passing to homotopy. The failure of \( m \) to be strictly commutative turns out to be a very interesting phenomenon, which is responsible for the existence of Steenrod operations.

In the above situation, the multiplication \( m \) is not commutative. However, it does induce a commutative multiplication after passing to cohomology. In fact, more is true: the map \( m \) satisfies a symmetry condition up to coherent homotopy. The following definitions allow us to make this idea precise:

**Definition 2.** Let \( V \) be an \( F_2 \)-module spectrum and \( n \geq 0 \) a nonnegative integer. The \( n \)th extended power of \( V \) is given by the homotopy coinvariants
\[
V^{\otimes n}_{h\Sigma_n}.
\]
This is a complex which we will denote by \( D_n(V) \).

**Remark 3.** In concrete terms, \( D_n(V) \) may be computed in the following way. Let \( M \) denote the vector space \( F_2 \), with the trivial action of \( \Sigma_n \). Choose a resolution
\[
\ldots \rightarrow P^{-1} \rightarrow P^0 \rightarrow M
\]
by free \( F_2[\Sigma_n] \)-modules. We let \( E\Sigma_n \) denote the complex \( P^* \). (We can think of \( E\Sigma_n \) as a contractible complex with a free action of \( \Sigma_n \).) The extended power \( D_n(V) \) of a complex \( V \) can then be identified with the ordinary coinvariants
\[
(V^{\otimes n} \otimes E\Sigma_n)_{\Sigma_n}.
\]

**Definition 4.** Let \( V \) be a complex. A symmetric multiplication on \( V \) is a map
\[
D_2(V) \rightarrow V.
\]

**Example 5.** If \( X \) is any topological space, then the cochain complex \( C^*(X; F_2) \) can be endowed with a symmetric multiplication. If \( X \) is equipped with a base point \(* \), then the reduced cochain complex \( C^*(X, *; F_2) \) also inherits a symmetric multiplication.

**Example 6.** Let \( X \) be an infinite loop space. Then the chain complex \( C_*(X; F_2) \) can be endowed with a symmetric multiplication.

Examples 5 and 6 are really special cases of the following:

**Example 7.** Let \( A \) be an \( E_\infty \)-algebra over the field \( F_2 \). Then \( A \) has an underlying \( F_2 \)-module spectrum, which is equipped with a symmetric multiplication.

Our goal in this lecture is to study the consequences of the existence of a symmetric multiplication on a complex \( V \).

**Notation 8.** Let \( n \) be an integer. We let \( F_2[-n] \) denote the complex which consists of a 1-dimensional vector space in cohomological degree \( n \), and zero elsewhere. Let \( e_n \) denote a generator for the \( F_2 \)-vector space \( H^n F_2[-n] \), so we have isomorphisms
\[
H^k F_2[-n] \cong \begin{cases} F_2 e_n & \text{if } k = n \\ 0 & \text{otherwise}. \end{cases}
\]

Our first goal is to describe the extended squares of complexes of the form \( F_2[-n] \). This is easy: we observe that \( F_2[-n]^{\otimes 2} \) is isomorphic to \( F_2[-2n] \), with the symmetric group \( \Sigma_2 \) acting trivially (since we are working in characteristic 2, there are no signs to worry about). Consequently, we can identify \( D_2(F_2[-n]) \) with the tensor product
\[
F_2[-2n] \otimes (E\Sigma_2)_{\Sigma_2}.
\]
The second tensor factor can be identified with the chain complex of the space \( B\Sigma_2 \simeq RP^\infty \). Consequently, we get canonical isomorphisms
\[
H^k(D_2(F_2[-n])) \cong H_{2n-k}(B\Sigma_2; F_2)e_{2n}.
\]
We now recall the structure of the homology and cohomology of the space $B\Sigma_2 \simeq \mathbb{R}P^\infty$. There is a (unique) isomorphism
\[ H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \simeq \mathbb{F}_2[t], \]
where the polynomial generator $t$ lies in $H^1(\mathbb{R}P^\infty; \mathbb{F}_2)$. We have a dual description of the homology $H_*(\mathbb{R}P^\infty; \mathbb{F}_2)$: this is just a one-dimensional vector space in each degree $m$, with a unique generator which we will denote by $x_m$.

**Definition 9.** Let $V$ be a complex, and let $v \in H^n V$, so that $v$ determines a homotopy class of maps
\[ \eta : \mathbb{F}_2[-n] \to V. \]
For $i \leq n$, we let
\[ Sq^i(v) \in H^{n+i} D_2(V) \]
denote the image of
\[ x_{n-i} \otimes e_{2n} \in H_{n-i}(\mathbb{R}P^\infty; \mathbb{F}_2) e_{2n} \simeq H^{n+i} D_2(\mathbb{F}_2[n]) \]
under the induced map
\[ D_2(\mathbb{F}_2[-n]) \xrightarrow{D_2(\eta)} D_2(V). \]
By convention, we will agree that $Sq^i(v) = 0$ for $i > n$.

If $V$ is equipped with a symmetric multiplication $D_2(V) \to V$, we let $Sq^i(v)$ denote the image of $Sq^i(v)$ under the induced map
\[ H^{n+i} D_2(V) \to H^{n+i} V. \]

The operations $Sq^i : H^* V \to H^{*+i} V$ are called the Steenrod operations, or Steenrod squares.

**Example 10.** Let $V$ be an $\mathbb{F}_2$-module spectrum equipped with a symmetric multiplication, and let $v \in H^n V$. Then $Sq^n(v) \in H^{2n} V$ is simply the image of $v \otimes v$ under the composite map
\[ V \otimes V \to D_2(V) \to V. \]
In other words, $Sq^n$ acts on $H^n V$ by simply “squaring” the elements with respect to the multiplication on $V$. This is why the operations $Sq^i$ are called “Steenrod squares”.

**Example 11.** Let $X$ be a topological space, and let $V = C^*(X; \mathbb{F}_2)$ be the cochain complex of $X$, equipped with its usual symmetric multiplication. Then Definition 9 yields operations
\[ Sq^i : H^n(X; \mathbb{F}_2) \to H^{n+i}(X; \mathbb{F}_2). \]
These are the usual Steenrod operations.

**Remark 12.** The operations $v \mapsto \overline{Sq}^i v$ completely account for the cohomology groups of any extended square $D_2(V)$. More precisely, let us suppose that $V$ is an $\mathbb{F}_2$-module spectrum, and that $\{v_i\}_{i \in I}$ is an ordered basis for $\pi_*(V)$, where $v_i \in H^{n_i} V$. Then the collection
\[ \{v_i v_j\}_{i < j} \cup \{Sq^n v_i\}_{n \leq n_i}, \]
is a basis for $\pi_* D_2(V)$. The proof of this is easy. Using the fact that $D_2$ commutes with filtered colimits, we can reduce to the case where only finitely many generators are involved. We then work by induction, using the formula
\[ D_2(V \oplus W) \simeq (V \oplus W)^{\otimes 2}_{h\Sigma_2} \simeq V^{\otimes 2}_{h\Sigma_2} \oplus (V \otimes W) \oplus W^{\otimes 2}_{h\Sigma_2} \]
to reduce to the case of a single basis vector. The result is then obvious.
**Proposition 13.** The Steenrod squares are additive operations. Let $V$ be a complex, and let $v, v' \in H^n V$. Then, for each integer $k$, we have

$$\overline{\text{Sq}}^k (v + v') = \overline{\text{Sq}}^k (v) + \overline{\text{Sq}}^k (v') \in H^{n+k} D_2(V).$$

In particular, if $V$ is equipped with a symmetric multiplication, we have

$$\text{Sq}^k (v + v') = \text{Sq}^k (v) + \text{Sq}^k (v') \in H^{n+k} V.$$

**Proof.** If $k > n$, then both sides are zero and there is nothing to prove. If $k = n$, then

$$\overline{\text{Sq}}^k (v + v') = (v + v')^2 = \overline{\text{Sq}}^k (v) + \overline{\text{Sq}}^k (v') + (vv' + v'v).$$

Since the multiplication map

$$V \otimes V \to D_2(V)$$

is commutative on the level of homotopy, we have $vv' + v'v = 2vv' = 0$.

Now suppose that $k < n$. By functoriality, it will suffice to treat the universal case where $V \simeq F[-n] \oplus F[-n]$. Using Remark 12, we observe that the canonical map

$$H^m D_2(V) \to H^m D_2(F_2[-n]) \times H^m D_2(F_2[-n])$$

is injective for $m < 2n$. We may therefore reduce to the case where either $v$ or $v'$ vanishes, in which case the result is obvious. \qed