A Pushout Square (Lecture 22)

In the last lecture we saw that the cohomology $H^* \mathcal{F}(n)$ of the free $E_\infty$-algebra on one generator was itself freely generated by one element, as an unstable algebra over the big Steenrod algebra $A^{\text{Big}}$. The Cartan-Serre theorem implies that the cohomology ring $H^* K(\mathbb{F}_2, n)$ is the free unstable $A$-module on one generator, in the same degree. This suggests a close relationship between $H^* \mathcal{F}(n)$ and $H^* K(\mathbb{F}_2, n)$. In fact, we can say more: there is a close relationship between the $E_\infty$-algebras $\mathcal{F}(n)$ and $C^* K(\mathbb{F}_2, n)$ for each $n \geq 0$.

To make this precise, we begin by observing that the canonical element $\nu \in H^n K(\mathbb{F}_2, n)$ gives rise to a map of $E_\infty$-algebras

$$f : \mathcal{F}(n) \to C^* K(\mathbb{F}_2, n).$$

Let $\mu$ denote the canonical generator of $H^* \mathcal{F}(n)$, so that $f$ carries $\mu$ to $\nu$.

The map $f$ is certainly not a homotopy equivalence. The target $H^* K(\mathbb{F}_2, n)$ is a module over the usual Steenrod algebra $A$, so that $Sq^0$ acts by the identity on $H^* K(\mathbb{F}_2, n)$. However, $Sq^0$ does not act by the identity on the cohomology of the left hand side. We therefore have

$$f(\mu - Sq^0 \mu) = f(\mu) - Sq^0 f(\mu) = \nu - Sq^0 \nu = 0,$$

so that $f$ fails to be injective on cohomology.

However, this turns out to be the only obstruction to $f$ being a homotopy equivalence. To make this precise, we observe that there is map $g : \mathcal{F}(n) \to \mathcal{F}(n)$, which is determined up to homotopy by the requirement that $g(\mu) = \mu - Sq^0 \mu \in H^n \mathcal{F}(n)$. The above calculation shows that $f \circ g$ carries $\mu$ to zero in $H^n K(\mathbb{F}_2, n)$. We therefore obtain a (homotopy) commutative diagram of $E_\infty$-algebras

$$\begin{array}{ccc}
\mathcal{F}(n) & \xrightarrow{g} & \mathcal{F}(n) \\
\downarrow & & \downarrow f \\
\mathbb{F}_2 & \xrightarrow{f} & C^* K(\mathbb{F}_2, n).
\end{array}$$

Our goal in this lecture is to prove:

**Theorem 1.** The above diagram is a homotopy pushout square in the category of $E_\infty$-algebras over $\mathbb{F}_2$.

In other words, the cochain complex $C^* K(\mathbb{F}_2, n)$ has a very simple presentation as an $E_\infty$-algebra over $\mathbb{F}_2$. It is “generated” by the tautological class $\nu \in H^n K(\mathbb{F}_2, n)$, and subject only to the “relation” that $\nu$ is fixed by $Sq^0$.

To prove Theorem 1, we need to understand homotopy pushouts in the world of $E_\infty$-algebras. We first recall the situation for ordinary commutative rings. Given a pair of commutative ring homomorphisms

$$A \leftarrow R \to B,$$

the pushout $A \bigsqcup_R B$ in the category of commutative rings is given by the relative tensor product $A \otimes_R B$. In the case of $E_\infty$-algebras, the situation is more or less identical. More precisely:
• Given an $E_\infty$-algebra $R$, there is a good theory of $R$-modules (or $R$-module spectra).
• Given any map $R \to A$ of $E_\infty$-algebras, we can regard $A$ as an $R$-module.
• Given an $E_\infty$-ring $R$, the collection of $R$-module spectra is endowed with a tensor product operation $(M, N) \mapsto M \otimes_R N$. (More traditionally, this is denoted by $M \wedge_R N$ and called the smash product over $R$).
• Given a pair of $E_\infty$-algebra maps

\[ A \leftarrow R \to B, \]

the homotopy pushout of $A$ and $B$ over $R$ in the setting of $E_\infty$-rings is again an $R$-algebra, and the underlying $R$-module is given by the tensor product $A \otimes_R B$.

Given these facts, we can restate Theorem 1. We have a canonical map

\[ \mathcal{F}(n) \otimes_{\mathcal{F}(n)} F_2 \to C^*K(F_2, n), \]

and we wish to show that this map is a homotopy equivalence. In other words, we wish to show that it induces an isomorphism after passing to cohomology. The cohomology of the right side is given by the Cartan-Serre theorem: $H^* K(F_2, n)$ can be identified with the polynomial ring on generators $\{Sq^i \nu\}$, where $i$ ranges over admissible positive sequences of excess $< n$. It therefore remains to compute the cohomology of the left hand side.

The calculation will be based on the following lemma:

**Lemma 2.** Let $R$ be an $E_\infty$-algebra over $F_2$, and let $M$ and $N$ be $R$-modules. Then $H^* M$ and $H^* N$ are modules over the cohomology ring $H^* R$. Suppose that $H^* M$ is free as a graded $H^* R$-module. Then the canonical map

\[ H^* M \otimes_{H^* R} H^* N \to H^*(M \otimes_R N) \]

is an isomorphism.

**Proof.** Choose elements $\{x_i \in H^n M\}$ which freely generate $H^* M$ as an $H^* R$-module. Each $x_i$ determines a map of $R$-modules $R[-n_i] \to M$. Adding these together, we obtain a map $\oplus R[-n_i] \to M$. By assumption this map induces an isomorphism on cohomology, and is therefore a homotopy equivalence. Thus, $M$ is a direct sum of free $R$-modules (in various degrees).

Let us say that an $R$-module $M$ is **good** if the canonical map

\[ H^* M \otimes_{H^* R} H^* N \to H^*(M \otimes_R N) \]

is an isomorphism. Both the left hand side and the right hand side above are functors of $M$, which commute with shifting and with the formation of direct sums. Therefore, to show that $\oplus R[-n_i]$ is good, it will suffice to show that $R$ is good. But this is clear, since

\[ H^* R \otimes_{H^* R} H^* N \simeq H^* N \simeq H^*(R \otimes_R N). \]

\[ \square \]

To prove Theorem 1, we will show that Lemma 2 applies: namely, that $H^* \mathcal{F}(n)$ is free when regarded as an $H^* \mathcal{F}(n)$-module via the map $g$. It then follows that we have an isomorphism

\[ H^*(\mathcal{F}(n) \otimes_{\mathcal{F}(n)} F_2) \simeq H^* \mathcal{F}(n) \otimes_{H^* \mathcal{F}(n)} F_2 = H^* \mathcal{F}(n)/I, \]

where $I$ is the ideal of $H^* \mathcal{F}(n)$ generated by the elements $g(x)$, where $x \in H^* \mathcal{F}(n)$ has positive degree.

In the last lecture, we proved that $H^* \mathcal{F}(n)$ is isomorphic to the free unstable $A_{\text{Alg}}$-module $F_{\text{Alg}}(n)$. It is therefore isomorphic to a polynomial ring on generators $\{Sq^i \mu\}$, where $I$ ranges over admissible sequences of excess $< n$. For every such sequence $I$, we let $X_I = g(Sq^i \mu) = Sq^i \mu - Sq^i Sq^0 \mu \in H^* \mathcal{F}(n)$. To complete the proof of Theorem 1, it will suffice to verify the following:
Proposition 3. The cohomology ring $H^* \mathcal{F}(n)$ is a polynomial ring on generators $\{X_I\}_{I \text{admissible of excess } < n}$ and $\{\text{Sq}^I \mu\}_{I \text{admissible and positive of excess } < n}$.

Proof. Let $\mathcal{J}$ denote the collection of all admissible sequences of integers of excess $< n$. We have a decomposition $\mathcal{J} = \mathcal{J}' \coprod \mathcal{J}''$, where $\mathcal{J}'$ consists of those sequences $(i_1, \ldots, i_k)$ such that $k > 0$ and $i_k < 0$. The complement $\mathcal{J}''$ has a further decomposition

$$\mathcal{J}'' = \mathcal{J}''(0) \coprod \mathcal{J}''(1) \coprod \cdots$$

where $\mathcal{J}''(m)$ consists of those sequence $(i_1, \ldots, i_k)$ which end with precisely $k$ zeroes. For each $I \in \mathcal{J}''(k)$, let $I^+ = \mathcal{J}''(k + 1)$ be the result of appending a zero to the sequence $I$. We have a decomposition

$$H^* \mathcal{F}(n) \simeq F_2[\text{Sq}^I \mu]_{I \in \mathcal{J}'} \otimes F_2[\text{Sq}^I \mu]_{I \in \mathcal{J}''}.$$ 

To complete the proof, it will suffice to show:

1. The polynomial ring $F_2[\text{Sq}^I \mu]_{I \in \mathcal{J}'}$ is also polynomial on the generators $\{X_I\}_{I \in \mathcal{J}'}$.

2. The polynomial ring $F_2[\text{Sq}^I \mu]_{I \in \mathcal{J}''}$ is also polynomial on the generators $\{X_I\}_{I \in \mathcal{J}''}$ and $\{\text{Sq}^I \mu\}_{I \in \mathcal{J}''(0)}$.

Assertion (2) follows immediately from the observation that $X_I = \text{Sq}^I \mu - \text{Sq}^{I^+} \mu$ for $I \in \mathcal{J}''$. We can divide the proof of (1) further into three steps:

1a) The map $\theta : F_2[X_I]_{I \in \mathcal{J}'} \to F_2[\text{Sq}^I \mu]_{I \in \mathcal{J}'}$ is well-defined. In other words, if $I \in \mathcal{J}'$, then $X_I$ belongs to $F_2[\text{Sq}^I \mu]_{I \in \mathcal{J}'}$.

1b) The map $\theta$ is injective.

1c) The map $\theta$ is surjective.

Assertion (1a) is an immediate consequence of the following:

Lemma 4. Let $I = (i_m, \ldots, i_1)$ be a sequence of integers with $i_1 < 0$. Then in $A^{Bp}$ we have an equality

$$\text{Sq}^I \text{Sq}^0 = \sum_{\alpha} \text{Sq}^{J_\alpha}$$

where each $J_\alpha$ is an admissible sequence of the form $(j_m, \ldots, j_0)$, where $j_0 < 0$.

Proof. We first apply the Adem relations to write

$$\text{Sq}^I \text{Sq}^0 = \sum_k (2k - i_1, -k - 1) \text{Sq}^k \text{Sq}^{i_1 - k}.$$ 

The coefficient $(2k - i_1, -k - 1)$ vanishes unless

$$\frac{i_1}{2} \leq k < 0.$$ 

We may therefore restrict our attention to those integers $k$ for which $i_1 - k < \frac{i_1}{2} < 0$, so the sequence $I'(k) = (i_m, \ldots, i_2, k, i_1 - k)$ ends with a negative integer.

Each $I'(k)$ can be rewritten as a sum of admissible monomials using the Adem relations. Let us analyze this process. Given a sequence $J = (j_m, \ldots, a, b, \ldots, j_0)$ with $a < 2b$, we have

$$\text{Sq}^J = \sum_k (2k - a, b - k - 1) \text{Sq}^{j_k},$$

where $J_k$ is obtained from $J$ by replacing $a$ by $b + k$ and $b$ by $a - k$. The coefficient $(2k - a, b - k - 1)$ vanishes unless $\frac{a}{2} \leq k < b$; in particular, we always have $a - k < \frac{a}{2} < b$. Thus, if the final entry in $J$ is negative, the final entry in $J_k$ will be negative. \qed
We now prove (1b). Recall that the cohomology ring $H^* \mathcal{F}(n) \simeq F_2[Sq^l \mu]_{l \in \mathbb{Z}}$ has a natural grading by rank, where $Sq^l \mu$ has rank $2^k$ for every sequence $I = (i_1, \ldots, i_k)$. This grading restricts to a grading on $F_2[Sq^l \mu]_{l \in \mathbb{Z}}$. We have an analogous grading on $F_2[X_I]_{l \in \mathbb{Z}}$, where we declare $\text{rk}(X_I) = 2^k$ if $I = (i_1, \ldots, i_k)$.

The map $\theta : F_2[X_I]_{l \in \mathbb{Z}} \rightarrow F_2[Sq^l \mu]_{l \in \mathbb{Z}}$ is not compatible with the gradings by rank. Instead we have

$$\theta(X_I) = Sq^l \mu - Sq^l Sq^0 \mu = Sq^l \mu + \text{higher rank}.$$  

We have an evident isomorphism $\theta' : F_2[X_I ]_{l \in \mathbb{Z}} \rightarrow F_2[Sq^l \mu]_{l \in \mathbb{Z}}$, given by $X_I \mapsto Sq^l \mu$. Let $x \in F_2[X_I]_{l \in \mathbb{Z}}$ be a nonzero element, and write $x$ as a sum $x = x_{k_0} + x_{k_1} + \ldots + x_{k_m}$ of homogeneous elements of ranks $k_0 < k_1 < \ldots < k_m$. Then we have

$$\theta(x) = \theta'(x) + \text{terms of rank } \leq k.$$  

In particular, $\theta(x) = 0$ implies $\theta'(x_{k_0}) = 0$. Since $\theta'$ is an isomorphism, we get $x_{k_0} = 0$, a contradiction. This completes the proof that $\theta$ is injective.

We now prove that $\theta$ is surjective. This is an immediate consequence of the following statement:

**Lemma 5.** Let $I = (i_k, \ldots, i_1)$ be a sequence of integers with $i_1 < 0$ (not necessarily admissible). Then $Sq^l \mu$ lies in the image of $\theta$.

**Proof.** We use descending induction on $i_1$. Observe that

$$Sq^l \mu = (Sq^l \mu - Sq^l Sq^0 \mu) + (Sq^l Sq^0 \mu) = \theta(X_I) + Sq^l Sq^0 \mu.$$  

It will therefore suffice to show that $Sq^l Sq^0 \mu$ belongs to the image of $\theta$. Using the Adem relations, we can write

$$Sq^l Sq^0 = \sum_k (2k - i_1, -k - 1) Sq^k$$  

with $I_k = (i_k, \ldots, i_2, k, i_1 - k)$. The coefficient $(2k - i_1, -k - 1)$ vanishes unless $\frac{i_1}{2} \leq k < 0$. This inequality forces

$$i_1 < i_1 - k \leq \frac{i_1}{2} < 0.$$  

Therefore $Sq^k$ belongs to the image of $\theta$ by the inductive hypothesis. 

**Corollary 6.** For each $n \geq 0$, the homotopy pullback square

$$\begin{array}{ccc}
K(F_2, n) & \rightarrow & * \\
\downarrow & & \downarrow \\
* & \rightarrow & K(F_2, n + 1)
\end{array}$$

of topological spaces determines a homotopy pushout square

$$\begin{array}{ccc}
C^* K(F_2, n) & \leftarrow & F_2 \\
\uparrow & & \uparrow \\
F_2 & \leftarrow & C^* K(F_2, n + 1)
\end{array}$$

of $E_\infty$-algebras.
Proof. Theorem 1 implies that $C^*K(F_2, n + 1)$ is freely generated by a single class $\nu$ in degree $(n + 1)$, subject to the single relation killing $\nu - Sq^0 \nu$. We can regard the homotopy pushout

$$F_2 \otimes_{C^*K(F_2, n+1)} F_2$$

as the suspension of $C^*K(F_2, n + 1)$ in the world of (augmented) $E_\infty$-algebras. Consequently, it has an analogous presentation as the free $E_\infty$-algebra generated by a class $\Sigma(\nu)$ in degree $n$, subject to a single relation killing $\Sigma(\nu - Sq^0 \nu)$. Since the Steenrod operation $Sq^0$ is stable, we can identify $\Sigma(\nu - Sq^0 \nu)$ with $\Sigma(\nu) - Sq^0 \Sigma(\nu)$. Applying Theorem 1 again, we can identify this suspension with $C^*K(F_2, n)$. It is easy to see that this identification is given by the map

$$F_2 \otimes_{C^*K(F_2, n+1)} F_2 \to C^*K(F_2, n)$$

described in the statement of Corollary 6. □