18.917 Topics in Algebraic Topology: The Sullivan Conjecture
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\[ p\)-adic Homotopy Theory (Lecture 27) \\

In this lecture we will continue to study the category \( \mathcal{S}_p^\vee \) of \( p\)-profinite spaces, where \( p \) is a prime number. Our main goal is to connect \( \mathcal{S}_p^\vee \) with the category of \( E_\infty \)-algebras over the field \( \overline{\mathbb{F}}_p \), following the ideas of Dwyer, Hopkins, and Mandell.

We begin with a brief review of rational homotopy theory. For any topological space \( X \), Sullivan showed how to construct a model for the rational cochain complex \( C^*(X; \mathbb{Q}) \) which admits the structure of a differential graded algebra over \( \mathbb{Q} \). The work of Quillen and Sullivan shows that the differential graded algebra \( C^*(X; \mathbb{Q}) \) completely encodes the “rational” structure of the space \( X \). For example, if \( X \) is a simply connected space whose homology groups \( H_i(X; \mathbb{Z}) \) are finitely generated, then the space \( X_\mathbb{Q} = \text{Map}(C^*(X; \mathbb{Q}); \mathbb{Q}) \) is a rationalization of \( X \): that is, there is a map \( X \to X_\mathbb{Q} \) which induces an isomorphism on rational homology. Here the mapping space \( \text{Map}(C^*(X; \mathbb{Q}); \mathbb{Q}) \) is computed in the homotopy theory of differential graded algebras over \( \mathbb{Q} \).

Our goal is to establish an analogue of this result, where we replace the field \( \mathbb{Q} \) by a field \( \mathbb{F}_p \) of characteristic \( p \). In this case, we cannot generally choose a model for \( C^*(X; \mathbb{F}_p) \) by a differential graded algebra (this is the origin of the existence of Steenrod operations). However, we can still view \( C^*(X; \mathbb{F}_p) \) as an \( E_\infty \)-algebra, and ask to what extent this \( E_\infty \)-algebra determines the homotopy type of \( X \). We first observe that \( C^*(X; \mathbb{F}_p) \) depends only on the \( p\)-profinite completion of \( X \). For any \( p\)-profinite space \( Y = \lim Y_\alpha \), we can define \( C^*(Y; \mathbb{F}_p) = \lim C^*(Y_\alpha; \mathbb{F}_p) \). If \( Y \) is the \( p\)-profinite completion of a topological space \( X \), then the canonical maps \( X \to Y_\alpha \) induce a map of \( E_\infty \)-algebras

\[ \theta : C^*(Y; \mathbb{F}_p) \simeq \lim C^*(Y_\alpha; \mathbb{F}_p) \to C^*(X; \mathbb{F}_p). \]

Since the the Eilenberg-MacLane spaces \( K(\mathbb{F}_p, n) \) are \( p\)-finite and represent the functor \( X \mapsto H^n(X; \mathbb{F}_p) \), we deduce that \( \theta \) is an isomorphism on cohomology.

Let \( k \) be any field of characteristic \( p \). Then, for every \( p\)-profinite space \( Y = \lim Y_\alpha \), we define

\[ C^*(Y; k) = C^*(Y; \mathbb{F}_p) \otimes_{\mathbb{F}_p} k \simeq \lim C^*(Y_\alpha; k). \]

**Warning 1.** If \( Y \) is the \( p\)-profinite completion of a space \( X \), then we again have a canonical map of \( E_\infty \)-algebras

\[ C^*(Y; k) \to C^*(X; k), \]

but this map is generally not an isomorphism on cohomology, since the Eilenberg-MacLane spaces \( K(k, n) \) are generally not \( p\)-finite.

Our goal is to prove the following:

**Theorem 2.** Let \( k \) be an algebraically closed field of characteristic \( p \). The functor

\[ X \mapsto C^*(X; k) \]

induces a fully faithful embedding from the homotopy theory of \( p\)-profinite spaces to the homotopy theory of \( E_\infty \)-algebras over \( k \).
We first need the following lemma:

**Lemma 3.** The functor $F$ defined by the formula

$$X \mapsto C^*(X; k)$$

carries homotopy limits of $p$-profinite spaces to homotopy colimits of $E_\infty$-algebras over $k$.

**Proof.** By general nonsense, it will suffice to prove that $F$ carries filtered limits to filtered colimits and finite limits to finite colimits.

For any category $\mathcal{C}$, the category $\text{Pro}(\mathcal{C})$ can be characterized by the following universal property: it is freely generated by $\mathcal{C}$ under filtered limits. In other words, $\text{Pro}(\mathcal{C})$ admits filtered limits, and if $\mathcal{D}$ is any other category which admits filtered limits, then functors from $\mathcal{C}$ to $\mathcal{D}$ extend uniquely (up to equivalence) to functors from $\text{Pro}(\mathcal{C})$ to $\mathcal{D}$ which preserve filtered limits. By construction, the functor $F$ is the unique extension of the functor $X \mapsto C^*(X; F_p)$ on $p$-finite spaces which carries filtered limits to filtered colimits.

To show that $F$ preserves finite limits to finite colimits, it will suffice to show that $F$ carries final objects to initial objects, and homotopy pullback diagrams to homotopy pushout diagrams. The first assertion is evident: $F(*) \cong k$ is the initial $E_\infty$-algebra over $k$. To handle the case of pullbacks, we note that every homotopy pullback square

$$
\begin{array}{ccc}
X' & \rightarrow & X \\
\downarrow & & \downarrow \\
Y' & \rightarrow & Y
\end{array}
$$

of $p$-profinite spaces is a filtered limit of homotopy pullback squares between $p$-finite spaces. We may therefore assume that the diagram consists of $p$-finite spaces, in which case we proved earlier that the diagram

$$
\begin{array}{ccc}
C^*(X'; F_p) & \leftarrow & C^*(X; F_p) \\
\uparrow & & \uparrow \\
C^*(Y'; F_p) & \leftarrow & C^*(Y; F_p)
\end{array}
$$

is a homotopy pushout square of $E_\infty$-algebras over $F_p$. The desired result now follows by tensoring over $F_p$ with $k$. \hfill \Box

**Lemma 4.** Let $\mathcal{K}$ be a collection of $p$-profinite spaces. Suppose that $\mathcal{K}$ contains every Eilenberg-MacLane space $K(F_p, n)$ and is closed under the formation of homotopy limits. Then $\mathcal{K}$ contains all $p$-profinite spaces $X$.

**Proof.** Every $p$-profinite space $X$ is a filtered homotopy limit of $p$-finite spaces. We may therefore assume that $X$ is finite. In this case, $X$ admits a finite filtration

$$X \simeq X_m \rightarrow X_{m-1} \rightarrow \ldots \rightarrow X_0 \simeq *$$

where, for each $i$, we have a homotopy pullback diagram

$$
\begin{array}{ccc}
X_{i+1} & \rightarrow & * \\
\downarrow & & \downarrow \\
X_i & \rightarrow & K(F_p, n_i).
\end{array}
$$

It follows by induction on $i$ that each $X_i$ belongs to $\mathcal{K}$. \hfill \Box
We now turn to the proof of Theorem 2. Fix a $p$-profinite space $Y$. For every $p$-profinite space $X$, we have a canonical map
\[ \theta_X : \text{Map}(Y, X) \to \text{Map}_K(C^*(X; k), C^*(Y; k)). \]
Let $\mathcal{K}$ denote the collection of all $p$-profinite spaces $X$ for which $\theta_X$ is a homotopy equivalence. Lemma 3 implies that both sides above are compatible with the formation of homotopy limits in $X$, so $\mathcal{K}$ is closed under the formation of homotopy limits. It will therefore suffice to show that every Eilenberg-MacLane space $K(F_p, n)$ belongs to $\mathcal{K}$. For each $i$, the map $\theta_{K(F_p, n)}$ induces a map
\[ H^{n-i}(Y; F_p) \simeq \pi_i \text{Map}(Y, K(F_p, n)) \to \pi_i \text{Map}_K(C^*(K(F_p, n); k), C^*(Y; k)) \simeq \pi_i \text{Map}_{F_p}(C^*(K(F_p, n); F_p), C^*(Y; k)); \]
we wish to show that these maps are isomorphisms.

We now specialize to the case $p = 2$, where we have described the cochain complex $C^*(K(F_p, n); F_p)$ as an $E_\infty$-algebra over $F_p$: namely, we have a pushout diagram of $E_\infty$-algebras
\[
\begin{array}{ccc}
\mathcal{I}(n) & \xrightarrow{u} & \mathcal{I}(n) \\
\downarrow & & \downarrow \\
F_p & \xrightarrow{\text{c}} & C^*(K(F_p, n); F_p)
\end{array}
\]
where the map $u$ classifies the cohomology operation $id - Sq^0$. It follows that we have a long exact sequence of homotopy groups
\[ \ldots \to H^{n-i-1}(Y; k) \to \pi_i \text{Map}_{F_p}(C^*(K(F_p, n); F_p), C^*(Y; k)) \to H^{n-i}(Y; k) \xrightarrow{id - Sq^0} H^{n-i}(Y; k) \to \ldots \]
To compute the homotopy groups of $\text{Map}_{F_p}(C^*(K(F_p, n); F_p), C^*(Y; k))$, we need to understand the cohomology ring $H^*(Y; k)$ as an algebra over the big Steenrod algebra $A_{\text{Big}}$. We observe that
\[ H^*(Y; k) \simeq H^*(Y; F_p) \otimes_{F_p} k. \]
The operation $Sq^0$ acts by the identity on the first factor, and by the Frobenius map $x \mapsto x^p$ on the field $k$. Since $k$ is algebraically closed, we have an Artin-Schreier sequence
\[ 0 \to F_p \to k \xrightarrow{v} k \to 0 \]
where $v$ is given by $v(x) = x - x^p$. It follows that the operation $id - Sq^0$ on $H^*(Y; k)$ is surjective, with kernel $H^*(Y; F_p)$. Thus the long exact sequence above yields a sequence of isomorphisms
\[ \pi_i \text{Map}_{F_p}(C^*(K(F_p, n); F_p), C^*(Y; k)) \simeq H^{n-i}(Y; F_p) \]
as desired.

Remark 5. The proof of Theorem 2 does not require that $k$ is algebraically closed, only that $k$ admits no Artin-Schreier extensions (that is, that any equation $x - x^p = \lambda$ admits a solution in $k$). Equivalently, it requires that the absolute Galois group $\text{Gal}(\overline{k}/k)$ have vanishing mod-$p$ cohomology.

Remark 6. Theorem 2 is false for a general field $k$ of characteristic $p$; for example, it fails when $k = F_p$. However, we can obtain a more general statement as follows. Suppose that $X$ is a $p$-profinite sheaf of spaces on the étale topos of $\text{Spec} k$; in other words, that $X$ is a $p$-profinite space equipped with a suitably continuous action $\sigma$ of the Galois group $\text{Gal}(\overline{k}/k)$. In this case, we get a Galois action on the cochain complex
\[ C^*(X; \overline{k}). \]
Using descent theory, we can extract from this an $E_\infty$-algebra of Galois invariants $C^*_\sigma(X; k)$, which we can regard as a $\sigma$-twisted version of the usual cochain complex $C^*(X; k)$ (these cochain complexes can be identified in the case where the action of $\sigma$ is trivial). The construction

$$(X, \sigma) \mapsto C^*_\sigma(X; k)$$

determines a functor from $p$-profinite sheaves on Spec $k$ to the category of $E_\infty$-algebras over $k$, and this functor is again fully faithful.