18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

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In this lecture we will combine some of our previous results to deduce a version of the Sullivan conjecture.

**Theorem 1.** Let $X$ be a finite-dimensional CW complex, $X^\vee$ its $p$-profinite completion, and $K$ a connected $p$-profinite space. Then the diagonal map

$$X^\vee \to (X^\vee)^K$$

is an equivalence of $p$-profinite spaces.

**Proof.** Let us say that a space $X$ is good if $X^\vee \to (X^\vee)^K$ is an equivalence. Since $p$-profinite completion preserves homotopy pushout squares (being a left adjoint) and $K$ is atomic in the $p$-profinite category, the collection of good spaces is stable under the formation of homotopy pushouts. We now show that every space $X$ of finite dimension $n$ is good, using induction on $n$. We have a homotopy pushout diagram

$$\bigsqcup S^{n-1} \longrightarrow \sk^{n-1} X$$

$$\bigsqcup D^n \longrightarrow X.$$

The inductive hypothesis guarantees that $\sk^{n-1} X$ and $\bigsqcup S^{n-1}$ are good. It will therefore suffice to show that $\bigsqcup D^n$ is good. But this coproduct is homotopy equivalent to a discrete topological space, which is obviously good. □

**Corollary 2.** Let $X$ be a finite dimensional CW complex, and $K$ a connected $p$-profinite space. Then every map $K \to X^\vee$ in the $p$-profinite category is homotopic to a constant map.

In the special case where $K = BG$, where $G$ is a finite $p$-group, we can identify $(X^\vee)^K$ with the homotopy fixed point set $(X^\vee)^{hG}$, where $G$ acts trivially on $X$. There is a more general form of Theorem 1 where we do not assume that the action of $G$ is trivial.

**Lemma 3.** Let $G$ be a finite $p$-group, and let $\mathcal{S}_p^\vee(G)$ denote the category of $p$-profinite spaces with an action of $G$. Then the functor

$$\mathcal{S}_p^\vee(G) \to \mathcal{S}_p^\vee$$

$$X \mapsto X^{hG}$$

preserves finite homotopy colimits.

**Proof.** We can identify $\mathcal{S}_p^\vee(G)$ with $\mathcal{S}_{p/\pi BG}$, and the formation of homotopy fixed points with the pushforward functor $f_*$, where $f : BG \to *$ is the projection. The desired result now follows from the observation that $BG$ is atomic. □
**Theorem 4.** Let $G$ be a finite $p$-group, $X$ a finite-dimensional $G$-CW complex, and $X^G$ the subcomplex of $G$-fixed points. Then the composite map

$$\phi: (X^G)\to (X^{hG})\to (X^h)^{hG}$$

is a homotopy equivalence of $p$-profinite spaces.

**Proof.** The space $X$ admits a filtration

$$X^G = Y_{-1} \subseteq Y_0 \subseteq \ldots \subseteq Y_n = X,$$

where $Y_j$ denotes the union of $X^G$ with the $j$-skeleton of $X$. We will prove by induction on $j$ that the conclusion of the theorem is valid for $Y_j$. The case $j = -1$ follows from Theorem 1. In the general case, we have a homotopy pushout diagram

$$\begin{array}{ccc}
\prod_{\alpha} S^{j-1} \times G/H_{\alpha} & \to & Y_{j-1} \\
\downarrow & & \downarrow \\
\prod_{\alpha} D^j \times G/H_{\alpha} & \to & Y_j,
\end{array}$$

where each $H_{\alpha}$ is a proper subgroup of $G$. Since $p$-profinite completion and passage to homotopy fixed points with respect to $G$ preserve homotopy pushout squares, we get a homotopy pushout square

$$\begin{array}{ccc}
((\prod_{\alpha} S^{j-1} \times G/H_{\alpha})^{hG}) & \to & (Y_{j-1}^{\vee})^{hG} \\
\downarrow & & \downarrow \\
((\prod_{\alpha} D^j \times G/H_{\alpha})^{\vee})^{hG} & \to & (Y_j^{\vee})^{hG},
\end{array}$$

of $p$-profinite spaces. By the inductive hypothesis, the upper right corner is equivalent to the $p$-profinite completion of $X^G$. It will therefore suffice to show that the $p$-profinite spaces in the left column are empty.

We will show that $Z = ((\prod_{\alpha} S^{j-1} \times G/H_{\alpha})^{\vee})^{hG}$ is empty; the same argument will show that $((\prod_{\alpha} D^j \times G/H_{\alpha})^{\vee})^{hG}$ is empty as well. The group $G$ has only finitely many proper subgroups $H$. We can therefore decompose $Z$ as a coproduct of spaces of the form

$$Z_H = ((\prod_{\alpha} S^{j-1} \times G/H)^{\vee})^{hG}.$$ 

It will therefore suffice to show that each $Z_H$ is empty. But $Z_H$ can be identified with

$$((\prod_{\alpha} S^{j-1} \times G/H)^{\vee})^{hG}.$$ 

We therefore have a map from $Z_H$ to the homotopy fixed set $(G/H)^{hG}$, which is empty because $H$ is a proper subgroup of $G$. \hfill \Box

**Remark 5.** We can formulate Theorem ?? as follows: the map $\phi$ identifies the homotopy fixed set $(X^\vee)^{hG}$ with the $p$-profinite completion of the actual fixed set $X^G$. 

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