Exercise 1. Let $F : \mathbb{R} \to \mathbb{R}$ is a $C^2$ map with uniformly bounded first and second derivatives. $F$ induces a map

$$\tilde{F} : C^0[0, 1] \to C^0[0, 1]$$

by composition; $\tilde{F}(u)$ is the function $t \mapsto F(u(t))$. Show that $\tilde{F}$ is a $C^1$ map. More generally let given a Banach space $B$ let $B^0 = C^0([0, 1], B)$ be the be space of continuous maps from $[0, 1]$ to $B$. Show that $B^0$ is a Banach space. If $F : B \to B$ is a $C^2$ map with uniformly bounded first and second derivatives, then the map induced by composition $\tilde{F}$ is $C^1$.

Exercise 2. Let $A : B \to B$ be a bounded linear operator. Consider the linear ODE in a Banach space

$$\frac{du}{dt} + Au = 0$$

with the initial condition $u(0) = v$. First show that the solution is given by

$$e^{-tA}v$$

where the time dependent operator $e^{-tA}$ is defined by showing the usual power series for the exponential is convergent in the Banach space of bounded linear operator from $B$ to itself. Let $B^0 = C^0([0, \epsilon], B)$ and $B^1 = C^1([0, \epsilon], B)$. Then we can view the differential equation as giving rise to a map

$$L : B^1 \to B^0 \times B$$

where

$$L(u) = \left( \frac{du}{dt}Au, u(0) \right).$$

Show that $L$ is invertible and indeed its inverse is given by the familiar formula

$$L^{-1}(u, v) = e^{-tA}v + \int_0^t e^{A(s-t)}u(s)ds$$
Exercise 3. The exercise uses the previous one to prove the existence and uniqueness theorem for first order ordinary differential equations. Let $B$ be a Banach space and let $X : B \rightarrow B$ be a $C^2$ map with bounded derivatives. We seek a solution to the differential question

$$\frac{du}{dt} + X(u) = 0$$

subject to the initial condition $u(0) = v$. Let $B^0 = C^0([0,\epsilon], B)$ and $B^1 = C^1([0,\epsilon], B)$. Then we can view the differential equation as given rise to a map

$$F : B^1 \rightarrow B^0 \times B$$

where

$$F(u) = (\frac{du}{dt} + X(u), u(0)).$$

Assuming the first exercise show that this a $C^1$ map. Show that The differential at $0$ is the map

$$D_0F(u) = (\frac{du}{dt} + D_0X(u), u(0))$$

which by the second exercise is invertible. Conclude from the this and the inverse function theorem the existence and unique ness theorem.

Exercise 4. Suppose that $V \rightarrow X$ is given as a subbundle of the trivial bundle $X \times \mathbb{R}^n \rightarrow X$ via a family of projections $\Pi$. Then the induced connection is $\Pi \circ d$ where $d$ denotes the ordinary deriviative. Given a local basis for $V$ find the connection matrix for the connection. Use this formula to find a connection matrix for $\gamma \rightarrow \mathbb{C}P^n$ be the tautogical bundle. (The tautological bundle sits inside the trivial $\mathbb{C}^{n+1}$ bundle.)