Lecture 2.

2 Smooth maps and the notion of equivalence

Let $X$ and $Y$ be smooth manifolds. A continuous map $f : X \to Y$ is called smooth if for all charts $(U, \phi)$ for $X$ and $(V, \psi)$ for $Y$ we have that the composition

$$\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \psi(V)$$

is smooth.

Two manifolds $X$ and $Y$ are called diffeomorphic if there is a homeomorphism $h : X \to Y$ so that $h$ and $h^{-1}$ are smooth.

3 Standard pathologies.

The condition that $X$ be Hausdorff and second countable does not follow from the existence of an atlas.

The line with two origins. Let $X$ be the quotient space of $\mathbb{R} \times \{0, 1\}$ by the equivalence relation $(t, 1) \equiv (t, 0)$ unless $t = 0$. Then $X$ is not Hausdorff, however $X$ admits an atlas with two charts. Let $U_i$ be the image of $\mathbb{R} \times \{i\}$ in $X$. These maps invert to give coordinates.

Remark 1. Actually non-Hausdorff spaces which satisfy all the other properties arise in real life for example in the theory of foliations or when taking quotients by non-compact group actions. More work is required to come up with a useful notions to replace that of manifolds in this context.

The long line. Let $S_\Omega$ denote the smallest uncountable totally ordered set. Consider the product $X = S_\Omega \times (0, 1]$ with dictionary order topology. Then give $X$ charts as follows. For $(\omega, t) \in X$ if $t \neq 1$ let $U_{(\omega, t)} = \{\omega\} \times (0, 1)$ and $\phi_{(\omega, t)} : U \to \mathbb{R}$ be given by $\phi_{(\omega, t)}(\omega, t) = t$. If $t = 1$ let $S(\omega)$ denote the successor of $\omega$. Set $U_{(\omega, 1)} = \{\omega\} \times (0, 1] \sup \{S(\omega)\} \times (0, 1)$ and

$$\phi_{(\omega, 1)}(\eta, t) = \begin{cases} t & \text{if } \eta = \omega \\ t + 1 & \text{if } \eta = S(\omega). \end{cases}$$

Exercise 5. Check that overlaps are smooth.
The collection \( \{ U_{(\omega, 1/2)} \}_{\omega \in S_\omega} \) is uncountable and consists of disjoint open sets, so \( X \) is not second countable.

Different charts

Consider \( \mathbb{R}_1 \) denote \( \mathbb{R} \) with the following charts \((\mathbb{R}, x)\) and \( \mathbb{R}_2 \) with the chart \((\mathbb{R}, x^3)\). Identity map \( \mathbb{R}_1 \to \mathbb{R}_2 \) is smooth but not \( \mathbb{R}_2 \to \mathbb{R}_1 \). \( \mathbb{R}_1 \) and \( \mathbb{R}_2 \) are diffeomorphic by the map \( x \mapsto x^3 \) thought of as a map from \( \mathbb{R}_1 \to \mathbb{R}_2 \).

These pathologies are simple problems to keep in mind when thinking about the definitions. There are far more subtle issues that arise. Given a topological manifold we can ask can carry an atlas, and if it carries an atlas how many non-diffeomorphic atlases does it carry. The first observation of this phenomenon is due to John Milnor who showed that the seven-sphere admits an atlas (with two charts!) which is not diffeomorphic to the standard differentiable structure. We’ll examine this example later in the course.