25 Differential forms and de Rham’s Theorem

25.1 The exterior algebra

Let $V$ be a finite dimensional vector space over the reals. The tensor algebra of $V$ is direct sum

$$\text{Ten}(V) = \mathbb{R} \oplus V \oplus V^\otimes 2 \ldots \oplus V^\otimes k \ldots$$

It is made into an algebra by declaring that the product of $a \in V^\otimes k$ and $b \in V^\otimes l$ is $a \otimes b \in V^\otimes (k+l)$. It is characterized by the universal mapping property that any linear map $V \rightarrow A$ where $A$ is an algebra over $\mathbb{R}$ extends to a unique map of algebras $\text{Ten}(V) \rightarrow A$.

The exterior algebra algebra is the quotient of exterior algebra by the relation

$$v \otimes v = 0.$$

The exterior algebra is denoted $\Lambda^\ast(V)$ or $\Lambda(V)$. It is customary to denote the multiplication in the exterior algebra by $(a, ) \mapsto a \wedge b$ If $v_1 \ldots v_k$ is a basis for $V$ then this relation is equivalent to the relations

$$v_i \wedge v_j = -v_j \wedge v_i \text{ for } i \neq j,$$

$$v_i \wedge v_i = 0.$$

Thus $\Lambda^\ast(V)$ has basis the products

$$v_{i_1} \wedge v_{i_2} \ldots v_{i_k}$$

where the indices run over all strictly increasing sequences of numbers between 1 and $n$.

$$1 \leq i_1 < i_2 < \ldots < i_k \leq n.$$

Since for each $k$ there are $\binom{n}{k}$ such sequences of length $k$ we have

$$\dim(\Lambda^\ast(V)) = 2^n.$$

$\Lambda^\ast(V)$ since the relation is homogenous the grading of the tensor algebra descends to a grading on the exterior algebra (hence the $\ast$).

We can apply this construction fiberwise to a vector bundle. The most important example is the cotangent bundle of a manifold $T^\ast X$ in which case we get the bundle of differential forms

$$\Lambda^\ast(T^\ast X) \text{ or } \Lambda^\ast(X).$$
We will denote the space of smooth sections of $\Lambda^* (X)$ by $\Omega^* (X)$. In local co-
dinates a typical element of $\Omega^* (X)$ looks like

$$
\omega = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \omega_{i_1 i_2 \ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}.
$$

Since the construction of $\Lambda^* (X)$ was functorial in the cotangent bundle these
bundles naturally pull back under diffeomorphism and if $f : X \to Y$ is any
smooth map there is natural map

$$
f^* : \Omega^* (Y) \to \Omega^* (X).
$$

The most important thing about differential forms is the existence of a natural
differential operator the exterior differential defined locally by the following rules

$$
d f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i
$$

$$
d \omega = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} d\omega_{i_1 i_2 \ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}.
$$

Notice that we can’t invariably define a similar operator on the tensor algebra.
If we have a one form

$$
\theta = \sum_{i=1}^{n} f_i dx^i
$$

and try to define

$$
D \theta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x^j} dx^j \otimes dx^i
$$

then when if we have new coordinates $y^1 \ldots y^n$ we have

$$
dx^i = \sum_{j=1}^{n} \frac{\partial x^i}{\partial y^j} dy^j
$$

and

$$
\theta = \sum_{m=1}^{n} g_m dy^m
$$

where

$$
g_m = f_i \frac{\partial x^i}{\partial y^m}
$$

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\[ D\theta = \sum_{i=1} \frac{\partial f_i}{\partial x^j} dx^j \otimes dx^i \]
\[ = \frac{\partial f_i}{\partial x^i} \frac{\partial x^i}{\partial y^l} dy^m \otimes dy^l \]
\[ = \frac{\partial f_i}{\partial y^k} \frac{\partial x^i}{\partial x^j} \frac{\partial x^j}{\partial y^l} dy^m \otimes dy^l \]
\[ = \frac{\partial f_i}{\partial y^m} \frac{\partial x^i}{\partial y^l} dy^m \otimes dy^l \]
\[ = \frac{\partial f_i}{\partial y^m} \frac{\partial x^i}{\partial y^l} dy^m \otimes dy^l \]
\[ = (\frac{\partial}{\partial y^m}(f_i \frac{\partial x^i}{\partial y^l}) - f_i \frac{\partial^2 x^i}{\partial y^m \partial y^l}) dy^m \otimes dy^l \]
\[ = \sum_{m=1}^n \frac{\partial g_l}{\partial y^m} dy^m \otimes dy^l - f_i \frac{\partial^2 x^i}{\partial y^m \partial y^l} dy^m \otimes dy^l. \]

Thus our definition depends on the choice of coordinates. Notice that when we pass to the exterior algebra this last expression vanishes that exterior derivative is well defined.

**Theorem 25.1.** \( d^2 = 0. \)

**Proof.** From the definition in local coordinates it suffices to check that \( d^2 = 0 \) on functions.
\[ d^2(f) = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j = 0 \]
since the \( f \) smooth so the matrix of second derivatives is symmetric. \( \square \)

**Proposition 25.2.**
\[ d(a \wedge b) = da \wedge b + (-1)^{\deg(a)} \wedge db. \]
Proof. The bilinearity of the wedge product implies that it suffices to check the result when
\[ a = f \, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}. \]

\[ \square \]

**Definition 25.3.** A cochain complex is a graded vector space \( C = \sum_{i=0}^{\infty} C_i \) together with a map \( d : C \to C \) so that \( dC_i \subset C_{i+1} \) and \( d^2 = 0 \). The cohomology groups of a cochain complex are defined to be
\[
H^i(C, d) = \ker(d : C^i \to C^{i+1})/\text{Ran}(d : C^{i-1} \to C^i)
\]