Lecture 8.

8 Connections

We motivate the introduction of connections in a vector bundle as a generalization of the usual directional derivative of functions on a manifold. Given a vector field $X$ and a function $f$ on a manifold $M$, its directional derivative is a new function as in equation (2). Thus we have a map

$$C^\infty(M; TM) \times C^\infty(M) \to C^\infty(M).$$

This map has the following properties.

$$X(fg) = fXg + gXf \quad (3)$$
$$\left(\alpha X + \beta Y\right)f = \alpha Xf + \beta Yf \quad (4)$$

where $X$ and $Y$ are smooth vector fields and $\alpha$, $\beta$, $f$ and $g$ are smooth functions.

If we try to generalize this to a directional derivative on sections of a vector bundle we would like a map

$$C^\infty(M; TM) \times C^\infty(M; E) \to C^\infty(M; E).$$

This map is using denoted

$$(X, s) \mapsto \nabla_X s$$

We can no longer multiply sections of a vector bundle but we can multiply sections of a vector bundle by functions. The appropriate generalization of the two rules about are

$$\nabla_X fs = f \nabla_X s + (Xf)s \quad (5)$$
$$\nabla_{\alpha X + \beta Y}s = \alpha \nabla_X s + \beta \nabla_Y f \quad (6)$$
9 Partitions of unity

Given an open cover, \( \{ U_\alpha | \alpha \in A \} \) of a topological space \( X \) we say that a collection of function \( \beta_\alpha : X \to \mathbb{R}_{\geq 0} \) is a partition of unity if

1. For all \( \alpha \in A \) \( \text{Support}(\beta_\alpha) \subset U_\alpha \)

2. The collection \( \{ \text{Support}(\beta_\alpha) | \alpha \in A \} \) is locally finite, that is to say for all \( x \in X \) there is a neighborhood of \( x \) meeting only finitely many of members of the collection.

3. For all \( x \in X \) we have \( \sum_{\alpha \in A} \beta_\alpha(x) = 1 \).

Smooth manifolds have smooth partitions of unity.

10 The Grassmanian is universal

We say that bundle is of finite type if there is a finite set of trivializations whose open sets cover. In this section we will prove the following theorem.

**Theorem 10.1.** Let \( E \to M \) be a vector bundle of finite type. Then for some \( N \) large enough there is a map

\[
f : M \to \text{Gr}_k(\mathbb{R}^N).
\]

**Proof.** Let \( \{(U_i, \tau_i) | i = 1, \ldots, m\} \) be a collection of trivializations so that the \( U_i \) cover. Write the trivializations as \( \tau_i(e) = (p(e), \phi_i(e)) \) as before. Choose a partition of unity \( \{\beta_i | i = 1, \ldots, m\} \) subordinate to the \( U_i \). Then define

\[
\Phi : E \to \mathbb{R}^{mk}
\]

by the formula

\[
\Phi(e) = (\beta_1(p(e))\phi_1(e), \beta_2(p(e))\phi_2(e), \ldots, \beta_m(p(e))\phi_m(e)).
\]

\( \Phi \) is well defined by the support condition on the partition of unity. \( \Phi \) is linear on each fiber of \( E \) as the \( \phi_i \) are. \( \Phi \) is injective on each fiber since for each \( b \in B \)
there is a $\beta_i$ with $\beta_i(b) \neq 0$. Thus for each point $b \in B$ we have that $\Phi^{-1}(p^{-1}(b))$ is a $k$-plane in $\mathbb{R}^{mk}$. So we can now define

$$f : B \rightarrow \text{Gr}_k(\mathbb{R}^{mk})$$

by

$$f(b) = \Phi(p^{-1}(b)).$$

**Exercise 6.** Check that this map is smooth. In other words write the map down in charts on the domain and range.

We claim that $f^*(\gamma_k)$ is isomorphic to $E$. Consider the map

$$\tilde{\Phi} : E \rightarrow B \times \gamma_k$$

given by

$$\tilde{\Phi}(e) = (p(e), (\Phi(p^{-1}(p(e))), \Phi(e))).$$

From the definition of $f$ this maps $E$ to $f^*(\gamma_k)$.

**Exercise 7.** Check that this is an isomorphism.