1. Symplectic Manifolds

Let \((M, \omega)\) be a symplectic manifold, i.e., a smooth manifold with nondegenerate closed 2-form \(\omega\).

**Example.** For \(X\) a smooth manifold, the cotangent bundle \(M = T^*X\) is a symplectic manifold. Specifically, given a chart \(U \subset X\) with coordinates \(x_1, \ldots, x_n\), we have a basis of \(T^*_pX\) given by \(dx_1, \ldots, dx_n\) and every \(\xi \in T^*_pX\) can be written as \(\sum \xi_i dx_i\). This gives us a map

\[
T^*X|_U \to \mathbb{R}^{2n}, (x, \xi) \mapsto (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)
\]

Let \(\alpha\) be the Liouville form defined by \(\sum \xi_i dx_i\) on each coordinate patch. It is well-defined as a 1-form on \(M\), and \(\omega = d\alpha = \sum d\xi_i \wedge dx_i\) is the desired symplectic form. Furthermore, given a diffeomorphism \(X_1 \to X_2\), we have an induced map

\[
F : T^*X_1 \to T^*X_2, (x, \xi) \mapsto (f(x), (d_x f)^{-1} \xi)
\]

which is a symplectomorphism (because \(\exists\) local coordinates in which \(f\) is the identity). Also, given \(h \in C^\infty(X, \mathbb{R})\), we have an associated symplectomorphism \(\tau_h : M \to M, (x, \xi) \mapsto (x, \xi + d_x h)\) since

\[
\tau_h^* \alpha = \alpha + dh \implies \tau_h^* \omega = \tau_h^*(d\alpha) = d\alpha + ddh = \omega
\]
as desired.

1.1. Submanifolds.

**Definition 1.** A submanifold \(W \subset (M, \omega)\) is symplectic if \(\omega|_W\) is symplectic (specifically, nondegenerate). This implies that \(T_p W \subset T_p M\) is a symplectic subspace \(\forall p\). \(L \subset (M, \omega)\) is Lagrangian if \(\omega|_L = 0\) and \(\dim L = \frac{1}{2} \dim M\).

**Example.** By our above construction, the 0-section \(X \hookrightarrow T^*X = M\) is a Lagrangian submanifold. Furthermore, sections of \(T^*X\) are graphs \(X_\mu = \{(x, \mu(x)) | x \in X\} \subset T^*X\) of 1-forms \(\mu \in \Omega^1(X, \mathbb{R})\); such a graph is Lagrangian iff \(d\mu = 0\), since denoting \(i_\mu(x) = (x, \mu(x))\), \(i_\mu^* \alpha = \mu \implies i_\mu^*(\omega) = i_\mu^*(d\alpha) = d(i_\mu^* \alpha) = d\mu\).

**Example.** For \(\Sigma^k \subset X^n\) a submanifold, define the conormal space to \(x \in \Sigma\) by

\[
N^*_x \Sigma = \{\xi \in T_x^*X | \xi|_{T_x \Sigma} = 0\}
\]

This gives us subbundle \(N^* \Sigma \subset T^*X|_{\Sigma}\) and a submanifold \(N^* \Sigma \subset T^*X\). For \(\Sigma = X\), we get the 0-section: for \(\Sigma = \{p\}\), we get the fiber \(T^*_p X\). By definition, \(\alpha|_{N^* \Sigma} = 0\), so \(N^* \Sigma\) is Lagrangian.

1.2. Symplectomorphisms and Lagrangian Submanifolds. Let \(\phi : (M_1, \omega_1) \to (M_2, \omega_2)\) be a diffeomorphism: we want to know whether \(\phi\) is a symplectomorphism as well, i.e. whether \(\phi^* \omega_2 = \omega_1\). Consider the graph \(\Gamma_\phi \subset M = M_1 \times M_2\). The latter space has one symplectic structure via \(\omega = \omega_1 \oplus \omega_2 = \pi_1^* \omega_1 + \pi_2^* \omega_2\), which is nondegenerate since

\[
\omega^{n_1 + n_2} = \binom{n_1 + n_2}{n_1} \pi_1^* \omega_1^{n_1} \wedge \pi_2^* \omega_2^{n_2}
\]

However, here we will consider the alternate symplectic structure given by \(\hat{\omega} = \pi_1^* \omega_1 - \pi_2^* \omega_2\).

**Proposition 1.** \(\phi\) is a symplectomorphism \(\iff \Gamma_\phi\) is Lagrangian.

**Proof.** \(\Gamma_\phi\) is the image of the embedding \(\gamma : M_1 \to M_1 \times M_2, p \mapsto (p, \phi(p))\), and \(\gamma^* \hat{\omega} = \gamma^* \pi_1^* \omega_1 - \gamma^* \pi_2^* \omega_2 = \omega_1 - \phi^* \omega_2\) is 0 \(\iff \Gamma_\phi\) is Lagrangian. \(\square\)
2. Hamiltonian Vector Fields

Let $M$ be a manifold.

**Definition 2.** An isotopy on $M$ is a $C^\infty$ map $\rho : M \times \mathbb{R} \to M$ s.t. $\rho_0 = \text{id}$ and $\forall t, \rho_t$ is a diffeomorphism.

Given an isotopy, we obtain a time-dependent vector field $v_t : p \mapsto \tfrac{d}{dt}\rho_t(q)|_{s=t}$ where $q = \rho_t^{-1}(p)$. We say that $\rho_t$ is the flow of $v_t$. Conversely, if $M$ is compact or $v_t$ is sufficiently "good", we can integrate to obtain the flow from the vector field. If $v$ is time-independent, we obtain a 1-parameter group $\rho_t = \exp(tv)$, with associated vector field $v$. Recall the Lie derivative $L_v\alpha = \tfrac{d}{dt}(\exp(tv)^*\alpha)|_{t=0}$.

**Proposition 2** (Cartan's Formula). $L_v\alpha = di_v\alpha + i_vd\alpha$.

If $(\rho_t)$ is generated by $(v_t)$ then $\tfrac{d}{dt}(\rho_t^*\alpha) = \rho_t^*(L_v\alpha)$.

Now, let $(M, \omega)$ be a symplectic manifold, $H : M \to \mathbb{R}$ a $C^\infty$ map. Then $dH \in \Omega^1(M) \implies \exists$ a unique vector field $X_H$ s.t. $i_{X_H}\omega = dH$, called the Hamiltonian vector field generated by $H$ (H itself is called the Hamiltonian function). Now, assume that $M$ is compact, or that the flow of $X_H$ is well-defined. Then we obtain an isotopy $\rho_t : M \to M$ of diffeomorphisms generated by $X_H$.

**Proposition 3.** $\rho_t$ are symplectomorphisms.

**Proof.** Note that $\tfrac{d}{dt}(\rho_t^*\omega) = \rho_t^*(L_{X_H}\omega)$ but $L_{X_H}\omega = di_{X_H}\omega + i_{X_H}d\omega = d^2H = 0$. Since $\rho_0$ is the identity, $\rho_t^*\omega = \omega$ for all $t$. \hfill $\square$

**Example.** For $\mathbb{R}^{2n}$ with coordinates $x_1, \ldots, x_n, p_1, \ldots, p_n$, the function $H(x, p) = \tfrac{1}{2}|p|^2 + V(x)$ has derivative $dH = \sum p_idp_i + \tfrac{\partial V}{\partial x_i}dx_i$. Thus, the associated vector field is $X_H = \sum -p_i\tfrac{\partial}{\partial x_i} + \tfrac{\partial V}{\partial x_i}\tfrac{\partial}{\partial p_i}$, giving us Hamilton's equations

\begin{equation}
\frac{dx_i}{dt} = -p_i = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = \frac{\partial V}{\partial x_i} = \frac{\partial H}{\partial x_i}
\end{equation}