SYMPLECTIC GEOMETRY, LECTURE 4

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1. HAMILTONIAN VECTOR FIELDS

Recall from last time that, for \((M, \omega)\) a symplectic manifold, \(H : M \to \mathbb{R}\) a \(C^\infty\) function, there exists a vector field \(X_H\) s.t. \(i_{X_H} \omega = dH\). Furthermore, the associated flow \(\rho_t\) of this vector field is an isotopy of symplectomorphisms.

Example. Consider \(S^2 \subset \mathbb{R}^3\) with cylindrical coordinates \((r, \theta, z)\) and symplectic form \(\omega = d\theta \wedge dz\) (\(\omega\) is the usual area form). Then setting \(H = z\) gives the vector field \(\frac{\partial}{\partial z}\): the associated flow is precisely rotation by angle \(t\).

Note also that the critical points of \(H\) are the fixed points of \(\rho_t\), and \(\rho_t\) preserves the level sets of \(H\), i.e.

\[
\frac{d}{dt} (H \circ \rho_t) = \frac{d}{dt} (\rho^*_t H) = \rho^*_t (L_{X_H} H) = \rho^*_t (i_{X_H} \omega(X_H)) = \rho^*_t (\omega(X_H, X_H)) = 0
\]

One can apply this to obtain the ordinary formula for conservation of energy.

**Definition 1.** \(X\) is a symplectic vector field if \(L_X \omega = 0\), i.e. \(i_X \omega\) is closed. \(X\) is Hamiltonian if \(i_X \omega\) is exact.

By Poincaré, we see that, locally, symplectic vector fields are Hamiltonian. Globally, we obtain a class \([i_X \omega] \in H^1(M, \mathbb{R})\).

Example. On \(T^2\), \(\frac{\partial}{\partial x}\) and \(\frac{\partial}{\partial y}\) are symplectic vector fields: since \(dy\) and \(dx\) are not exact, \(\frac{\partial}{\partial x}\) and \(\frac{\partial}{\partial y}\) are not Hamiltonian.

Now consider time-dependent Hamiltonian functions, i.e. \(C^\infty\) maps \(\mathbb{R} \times M \to \mathbb{R}, (t, x) \mapsto H_t(x)\). Let \(\text{Ham}(M, \omega)\) denote the space of Hamiltonian diffeomorphisms on \(\omega\), i.e. the set of diffeomorphisms \(\rho\) s.t. \(\exists H_t\) with corresponding flow \(\rho_t\) satisfying \(\rho_1 = \rho\).

**Remark.** The Arnold conjecture states that for \(M\) compact, \(\phi \in \text{Ham}(M, \omega)\) with nondegenerate \(\text{Fix}(\phi)\) (i.e. at a fixed point \(p\), \(d\phi(p) - \text{id}\) is invertible),

\[
\#\text{Fix}\phi \geq \sum \dim H^i(M)
\]

This statement is false for non-Hamiltonian vector fields, as seen in the case of \(\frac{\partial}{\partial x}\) on a torus.

We can measure the difference between symplectomorphisms and Hamiltonian symplectomorphisms via the flux function

\[
\text{Flux}(\rho_t) = \int_0^1 [i_{X_t} \omega] dt \in H^1(M, \mathbb{R})
\]

In general, the flux depends on the homotopy class of the path from the identity to \(\rho_1\).

**Remark.** The Flux conjecture concerns the integral of the flux on \(\pi_1\text{Symp}(M, \omega)\), i.e. the nature of

\[
(\text{Flux}, \pi_1\text{Symp}(M, \omega)) \subset H^1(M, \mathbb{R})
\]

Geometrically, for \(\gamma : S^1 \to M\) a loop, let \(\gamma_t = \rho_t \circ \gamma : S^1 \to M\) be the image of \(\gamma\) under \(\rho\) and define \(\Gamma : [0,1] \times S^1 \to M\) by \((t, s) \mapsto \gamma_t(s)\).

**Problem.** \(\langle \text{Flux}(\rho_t), [\gamma]\rangle = \text{Area}(\Gamma) = \int_{[0,1] \times S^1} \Gamma^* \omega\).
2. Moser’s Theorem

One can ask whether, for a given manifold $M$, two symplectic structures $\omega_0, \omega_1$ are equivalent, i.e. whether there is a symplectomorphism $M \to M$ which pulls back one to the other. In general, $[\omega_0] = [\omega_1]$ does not imply that the two structures are symplectomorphic. To study this question further, we give other notions of equivalence.

**Definition 2.** Two forms $\omega_0, \omega_1$ are deformation equivalent if $\exists \{\omega_t\}_{t \in [0,1]}$ a continuous family of symplectic forms, and isotopic if there is such a family with $[\omega_t]$ constant in $H^2(M, \mathbb{R})$.

**Remark.** There exist pairs of symplectic forms with the same cohomology class which are not deformation equivalent, as well as pairs which are deformation equivalent but not isotopic (in dimension $\geq 6$).

Let $M$ be a compact manifold with $\omega_0, \omega_1$ isotopic symplectic forms (i.e. $\exists \omega_t$ as above with each $\omega_t$ nondegenerate).

**Theorem 1** (Moser). $\exists$ an isotopy $\rho_t : M \to M$ s.t. $\rho_t^* \omega_t = \omega_0$.

That is, $(M, \omega_0)$ and $(M, \omega_1)$ are symplectomorphic.

**Proof.** (This technique is known as Moser’s trick.) By assumption, $[\omega_t]$ is independent of $t$, i.e. $[d\omega_t] = 0$. Thus, $\exists \alpha_t$ a 1-form s.t. $\frac{d\omega_t}{dt} = -d\alpha_t$; moreover, we can choose this $\alpha_t$ smoothly w.r.t. $t$ (via the Poincaré lemma). Since $\omega_t$ is nondegenerate, $\exists X_t$ s.t. $i_{X_t} \omega_t = \alpha_t$. Moreover, since $M$ is compact, we have a well-defined flow $\rho_t$ of $X_t$. Now,

$$(5) \quad \frac{d}{dt}(\rho_t^* \omega_t) = \rho_t^*(L_{X_t} \omega_t) + \rho_t^* \left( \frac{d\omega_t}{dt} \right) = \rho_t^*(dX_t \omega_t + \frac{d\omega_t}{dt}) = 0$$

Since $\rho_0$ is the identity, we have our desired isotopy. \qed

**Example.** For symplectic forms $\omega_0, \omega_1$ with $[\omega_0] = [\omega_1]$, consider the family $\omega_t = t \omega_0 + (1-t) \omega_1$. By the above, if this family is nondegenerate, the two forms are symplectomorphic. In general, there is no reason for this to be true: in dimension 2, it always is. More generally, this follows from compatibility with almost-complex structures.

**Theorem 2** (Darboux). For $(M, \omega)$ symplectic, $p \in M$, $\exists U \ni p$ with a coordinate system $(x_1, y_1, \ldots, x_n, y_n)$ s.t. $\omega|_U = \sum dx_i \wedge dy_i$.

**Proof.** $(T_p M, \omega_p)$ has a standard basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$, so there exist local coordinates $(x_1, y_1, \ldots, x_n, y_n)$ s.t. $\omega_p = \sum dx_i \wedge dy_i$. On a neighborhood $U$ of $p$, we obtain two symplectic forms: $\omega$ and the standard form. The family $\omega_t = (1-t) \omega_0 + t \omega_1$ is one of closed forms: since nondegeneracy is an open condition, we can shrink our neighborhood to assure that $\omega_t$ is nondegenerate for each $t$ on some $U' \ni p$. Thus, $\exists \alpha \in \Omega^1(U)$ s.t. $\omega_t - \omega_0 = -d\alpha$. Subtracting a constant, we can assume $\alpha_p = 0$. Let $v_t$ be the vector field on $U$ s.t. $i_{v_t} \omega_t = \alpha$. Then $\exists U'' \ni p$ s.t. its flow $\rho_t$ is defined $\forall t$. By the Moser’s trick, we find that $\rho_t^* \omega_1 = \omega_0$, implying that the symplectic form is indeed standard after composing our chosen coordinates with $\rho_1$. \qed