SYMPLECTIC GEOMETRY, LECTURE 6

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1. Applications

(1) The work done last time gives us a new way to look at $T_d\text{Symp}(M,\omega)$ (using $C^1$-topology, wherein $f_i : X \to Y$ converges to $f$ iff $f_i \to f$ uniformly on compact sets and same for $df_i : TX \to TY$.

Now, $f \in \text{Symp}(M,\omega)$ gives a graph map $\text{graph}(f) = \{(x, f(x))\} \subset (M \times M, \text{pr}_1 \omega - \text{pr}_2 \omega)$ which is a Lagrangian submanifold. If $f$ is $C^1$-close to the identity map, then graph$(f)$ is $C^1$-close to the diagonal $\Delta = \{(x, x)\} \subset (M \times M, \text{pr}_1 \omega - \text{pr}_2 \omega)$ (i.e. the graph of the identity map). By Weinstein, a tubular neighborhood of $\Delta$ is diffeomorphic to $U_0 \subset (T^* M, \omega_{\gamma^*})$, and the graph of $f$ gives a section ($C^1$-close to the zero section), i.e. the graph of a $C^1$-small $\mu \in \Omega^1(M)$. The fact that its graph is Lagrangian implies that $\mu$ is closed, i.e. $d\mu = 0$. Thus, we have an identification $T_d\text{Symp}(M,\omega) \cong \{\mu \in \Omega^1 | d\mu = 0\}$ with $C^1$ topologies.

(2) Theorem 1. For $(M,\omega)$ compact, if $H^1(M,\mathbb{R}) = 0$, then every symplectomorphism of $M$ which is $C^1$ close to the identity has $\geq 2$ fixed points.

Theorem 2. For $(M,\omega)$ symplectic, $X \subset (M,\omega)$ compact and Lagrangian, if $H^1(X,\mathbb{R}) = 0$, then every Lagrangian submanifold of $M$ which is $C^1$ close to $X$ intersects $X$ in $\geq 2$ points.

The first theorem follows from the second, using the diagonal embedding $\Delta \subset M \times M$. To see the second theorem, note that $H^1(X) = 0$ implies that, given any graph $Y = \text{graph}(\mu)$ $C^1$-close to $X$ with $d\mu = 0$, we have $\mu = dh$ for some $h : X \to \mathbb{R}$. Since such an $h$ must have at least 2 critical points, there are at least 2 points at which $\mu = 0$, i.e. points at which $Y$ intersects $X$.

2. Arnold Conjecture

Arnold’s conjecture: Let $(M,\omega)$ be compact, $f \in \text{Ham}(M,\omega)$ the time 1 flow of $X_{H_t}$ for $H_t : M \to \mathbb{R}$ a 1-periodic Hamiltonian $(H : M \times \mathbb{R} \to \mathbb{R}$ smooth with $H_{t+1} = H_t$). Then the number of fixed points of $f$ is at least the minimal number of critical points of a smooth function on $M$. Moreover, assume the fixed points of $f$ are nondegenerate, i.e. if $f(x) = x$ then det $(df_x - \text{id}) \neq 0$. Then $\#\text{Fix}(f)$ is at least the minimal number of critical points of a Morse function on $M$, which in turn is $\geq \sum_i \dim H^i(M)$.

Remark. The last inequality follows from classical Morse theory. Given a Morse function $f$ on a manifold $M$ (equipped with a Riemannian metric satisfying the Morse-Smale condition), we have the Morse complex $C^i$ generated by critical points of index $i$, and the Morse differential $d : C^i \to C^{i+1}$ which counts gradient trajectories between critical points. Then $H^*(C^*, d) \cong H^*(M)$, so $\#\text{Fix}(f) = \sum \dim C^i \geq \sum \dim H^i$.

The case where $H_t = H$ is independent of $t$ is easy: if $p$ is a critical point of $H$ then $X_H(p) = 0$ so the flow $f$ fixes $p$. The general case was proved by Conley-Zehnder, Floer, Hofer-Salamon, Ono, Fukaya-Ono, Li-Tian, ... using Floer homology. Floer homology is formally the $\infty$-dimensional Morse theory of a functional on a covering of the loop space, $\Omega M = \{\gamma : S^1 \to M$ contractible + homotopy class of disc with $\partial D = \gamma\}$:

\[ \mathcal{A}_H : \Omega M \to \mathbb{R}, \quad \mathcal{A}_H(\gamma) = -\int_{D^2} u^*\omega - \int_{S^1} \int_{S^1} H(t, \gamma(t)) dt \]

where the first term involves $u : D^2 \to M$ with $u(\partial D) = \gamma$ in the given homotopy class.
Given $v : S^1 \to \gamma^*TM$ (a vector field along $\gamma$), the differential of $A_H$ is given by

$$DA_H(\gamma)(v) = -\int_{S^1} \omega(v(t), \dot{\gamma}(t)) \, dt - \int_{S^1} dH_t(\gamma(t))(v(t)) \, dt = \int_{S^1} (i_{\dot{\gamma}(t)}\omega - dH_t)(v(t)) \, dt.$$ 

Since $dH_t = i_{X_t}\omega$, this vanishes $\forall v$ if and only if $\dot{\gamma}(t) = X_t(\gamma(t))$, i.e. $\gamma$ is a periodic orbit of the flow. Hence critical points of $A_H$ correspond to fixed points of $f$. Moreover, formally gradient trajectories of $A_H$ correspond to solutions $u : \mathbb{R} \times S^1 \times M, (s, t) \mapsto u(s, t)$ of the PDE

$$(2) \quad \frac{\partial u}{\partial s} + J(u) \left( \frac{\partial u}{\partial t} - \nabla H_t(u) \right) = 0.$$