1. Chern Classes

Let $E \to M$ be a complex vector bundle, $\nabla$ a connection on $E$. Recall that we obtain the Chern classes of $E$ via $c(E) = \sum c_i(E) = \det (I + \frac{1}{2\pi} R^\nabla)$.

**Proposition 1.** $M$ compact and oriented $\implies c_r(E) = c(E) \in H^{2r}(M, \mathbb{Z})$.

Let $s$ be a section transverse to the zero section. Let $Z = s^{-1}(0)$ be its zero set: then

$$[Z] \in H_{2n-2r}(M) \implies PD([Z]) \in H^{2r}(M)$$

is the Euler class of $E$.

1.1. Chern Classes of Line bundles. We now restrict to understanding the first Chern class of a line bundle. If $M$ is compact, this is precisely the Euler class. Now, consider a closed, oriented surface $\Sigma$: any section vanishes at finitely many points, giving us a well-defined degree by counting these points (with sign). Moreover, we have that $c_1(L) \in H^2(L, \mathbb{Z}) \cong \mathbb{Z}$ is precisely the class s.t. $c_1(L)[\Sigma] = \deg L$. Cut $\Sigma$ into two parts $U \cup D^2$, where $U = \sqrt{S^1}$ holds all the non-trivial loops. Any complex bundle over $S^1$ is trivial, so $L$ is trivial over both $U$ and $D^2$. To obtain $L$ from $L|_U$ and $L|_{D^2}$, we need to identify $L|_{\partial U} \cong \mathbb{C} \times S^1 \to L|_{\partial D^2} \cong \mathbb{C} \times S^1$. This corresponds to a map $S^1 \to \mathbb{C}^*$ modulo homotopy, i.e. an element of $\pi_1(S^1) \cong \mathbb{Z}$. This is again deg $L$.

**Remark.** Alternatively, since $L$ is trivial over $D^2$ and $U$, we have a non-vanishing section $s$ of $L|_U$. The Chern class of $L$ measures why this section cannot be extended to all of $\Sigma$. Specifically, the Chern class corresponds to the boundary map $\frac{s}{|s|} : \partial D^2 = S^1 \to S^1$.

1.2. Properties of Chern Classes. Let $c(E) = \sum c_i(E)$ denote the total Chern class of $E$ (with $c_0(E) = 1$).

1. $c(E \oplus F) = c(E) \cup c(F)$.
2. For $f : X \to M$ a smooth map giving a commutative square

$$
\begin{array}{ccc}
 f^* E & \longrightarrow & E \\
 \downarrow & & \downarrow \\
 X & \longrightarrow & M \\
\end{array}
$$

where $f^* E = \{ (x, v) \in X \times E | f(x) = \pi(v) \}$, we have $c(f^*(E)) = f^*(c(E))$. By the splitting principle, for any $E \to M$, $\exists f : X \to M$ s.t. in the above square, $f^*$ is injective on cohomology, and $f^* E$ splits as a sum of line bundles.

One can define the Chern classes via these properties along with the definition of the first Chern class of a line bundle. Our definition of Chern classes (i.e. via the curvature $R^\nabla$) also satisfies these properties.

1. Given bundles $E, F$ with connections $\nabla^E, \nabla^F$, the connection on the direct sum is precisely $\nabla^{E \oplus F}(s, t) = (\nabla^E(s), \nabla^F(t))$, implying that the curvature is $R^{E \oplus F} = R^E \oplus R^F$ as desired.
2. Note that, if $s$ is a local section of $E$ near $f(x)$, then $s \circ f$ is a local section of $f^* E$ near $x$. By the definition of the pullback connection, $\nabla^{f^* E}(f^*(s)) = f^*(\nabla^E s)$. Via the definition of curvature, we see that $f^*(R^\nabla) = R^{f^* \nabla}$ as well, implying the desired pullback property.

**Remark.** $c_1(L) \in H^2(M, \mathbb{Z})$ completely classifies $\mathbb{C}$-line bundles. Moreover, it defines a group isomorphism between the set of line bundles over $M$ under $\otimes$ with $H^2(M, \mathbb{Z})$. To see this, recall that a line bundle is precisely
a collection of local trivializations \( \{ f_\alpha : L|_{U_\alpha} \cong U_\alpha \times \mathbb{C} \} \) with attaching maps \( g_{\alpha,\beta} \in C^\infty(U_\alpha \cap U_\beta, \mathbb{C}^*) \) satisfying the cocycle condition

\begin{equation}
(3) \quad g_{\alpha,\beta}g_{\beta,\gamma}g_{\gamma,\alpha} = 1
\end{equation}

on \( U_\alpha \cap U_\beta \cap U_\gamma \). This corresponds precisely with the Cech cohomology on \( M \), where \( \{ g_{\alpha,\beta} \} \) is a 1-cocycle. In this description, \( c_1 \) is the connecting map in the long exact sequence

\begin{equation}
(4) \quad \cdots \to 0 = H^1(M, \mathbb{C}) \to H^1(M, \mathbb{C}^*) \to c_1 H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{C}) = 0 \to \cdots
\end{equation}

associated to the short exact sequence of sheaves \( 0 \to \mathbb{C} \to \mathbb{C} \to \mathbb{C}^* \to 0 \) where \( \mathbb{C}, \mathbb{C}^* \) are the sheaves of \( \mathbb{C}^\infty \) functions with values in \( \mathbb{C}, \mathbb{C}^* \). One can also see directly the fact that \( c_1(L \otimes L') = c_1(L) + c_1(L') \) using the definition of the tensor product connection \( \nabla_{L \otimes L'} = \nabla_L \otimes \text{id} + \text{id} \otimes \nabla_{L'} \).

Now, for \((M, \omega)\) a symplectic manifold, \( J \) a compatible almost-complex structure, \((TM, J)\) is a complex vector bundle, with \( c_2(TM) \in H^2(M, \mathbb{Z}) \). Since the RHS is discrete, we get an invariant of the almost-complex structure up to deformation, and since the space of compatible \( J \)'s is connected, the complex isomorphism class of \((TM, J)\) is uniquely determined. Explicitly, if \( J_t \) is a family of complex structures on \( E \), the map \( \phi : v \mapsto \frac{1}{2}(v - J_t J_{t_0} v) \) is a complex isomorphism from \((E, J_{t_0})\) to \((E, J_t)\) since

\begin{equation}
(5) \quad \phi(J_{t_0} v) = \frac{1}{2}(J_{t_0} v + J_t v) = J_t(\frac{1}{2}(v - J_t J_{t_0} v)) = J_t \phi(v)
\end{equation}

Thus, \( c_2(TM, J) \) is independent of the choice of almost-complex structure (it is even an invariant of the deformation class of \( M \)): for instance, \( c_2(TM) \in H^2(M, \mathbb{Z}) \cong \mathbb{Z} \) is an invariant of the manifold (the Euler characteristic).

\textbf{Remark}. For \( 1 \leq j \leq n - 1 \), \( c_j \) does depend on the choice of symplectic structure, however: there exists a 4-manifold \( M \) with symplectic forms \( \omega_1, \omega_2 \) s.t. \( c_1(TM, \omega_1) \neq c_1(TM, \omega_2) \).