Recall that we were in the midst of elliptic operator analysis of the Laplace-deRham operator $\Delta = (d + d^*)^2$. We claimed that $\Delta$ was an elliptic operator, i.e. it has an invertible symbol $\sigma(\xi) = -|\xi|^2 \text{id}$. We stated that a differential operator $L : C^\infty(E) \to C^\infty(F)$ of order $k$ extends to a map $L_s : W^s(E) \to W^{s-k}(F)$.

**Definition.** For $L : \Gamma(E) \to \Gamma(F)$ a differential operator, $P : \Gamma(F) \to \Gamma(E)$ is called a parametrix (or pseudoinverse) if $L \circ P - \text{id}_E$ and $P \circ L - \text{id}_F$ are smoothing operators, i.e. they extend continuously to $W^s(E) \to W^{s+1}(E)$.

The following results can be found in Wells’ book.

**Theorem 1.** Every elliptic operator has a pseudoinverse.

**Corollary 1.** $\xi \in W^s(E), L \text{ elliptic, } L\xi \in C^\infty(F) \implies \xi C^\infty(E)$.

**Theorem 2.** $L \text{ elliptic } \implies L_s$ is Fredholm, i.e. Ker $L_s, \text{Coker } L_s$ are finite dimensional, Im $L_s$ is closed, and Ker $L_s = \text{Ker } L \subset C^\infty(E)$.

**Theorem 3.** $L \text{ elliptic, } \tau \in (\text{Ker } L^*)^\perp = \text{Im } L \subset C^\infty(F) \implies \exists \xi \in C^\infty(E)$ s.t. $L\xi = \tau$ and $\xi \perp \text{Ker } L$.

**Theorem 4.** $L \text{ elliptic, self-adjoint } \implies \exists H_L, G_L : C^\infty(E) \to C^\infty(E)$ s.t.

1. $H_L$ maps $C^\infty(E) \to \text{Ker } (L)$,
2. $L \circ G_L = G_L \circ L = \text{id} - H_L$,
3. $G_L, H_L$ extend to bounded operators $W^s \to W^s$, and
4. $C^\infty(E) = \text{Ker } L \oplus_{1, L^2} \text{Im } (L \circ G_L)$.

We now return to the case of $\Delta = (d + d^*)^2$ on a compact manifold.

**Corollary 2.** $\exists G : \Omega^k \to \Omega^k$ and $H : \Omega^k \to \mathcal{H}^k = \text{Ker } \Delta$ s.t. $G\Delta = \Delta G = \text{id} - H$ and $\text{Im } (G\Delta) = (\mathcal{H}^k)^\perp$.

**Corollary 3.** $\Omega^k = \mathcal{H}^k \oplus_{1, L^2} \text{Im } d + \text{Im } d^*$.

**Remark.** Every $\alpha \in \Omega^k$ decomposes as $\alpha = H\alpha + d(d^*G\alpha) + d^*(dG\alpha)$.

Using this decomposition, we immediately obtain the theorem

**Theorem 5** (Hodge). For $M$ a compact, oriented Riemannian manifold, every cohomology class has a unique harmonic representative.

From now on, $M$ is a compact, Kähler manifold, with the Hodge $*$ operator on $\Omega^*(M)$ extended $\mathbb{C}$-linearly to $\mathbb{C}$-valued forms.

**Proposition 1.** $*$ maps $\bigwedge^{p,q} \to \bigwedge^{n-q, n-p}$.

**Proof.** Consider the standard orthonormal basis of $V = T_x^* M$ given by $\{x_1, y_1, \ldots, x_n, y_n\}$ with $Jx_j = y_j$ and $z_j = x_j + iy_j$ giving the basis for $\Lambda^{1,0}$. Now, write any form $\alpha$ as a linear combination of

$$\alpha_{A,B,C} = \prod_{j \in A} z_j \wedge \prod_{j \in B} \bar{z}_j \wedge \prod_{j \in C} z_j \wedge \bar{z}_j$$

where $A, B, C \subset \{1, \ldots, n\}$ are disjoint subsets. That is, $A$ is the set of indices which contribute purely holomorphic terms of $\alpha$, $B$ is the set of indices which contribute purely anti-holomorphic terms to $\alpha$, and $C$ is the set of indices which contribute both. One can show that

$$(\alpha_{A,B,C})^* = \text{id}^{-a-b}(-1)^{\frac{1}{2}k(k+1)+c(2i)^k-n} \alpha_{A,B,C}$$

where $C' = \{1, \ldots, n\} \setminus (A \cup B \cup C), a = |A|, b = |B|, c = |C|$, and $k = \deg \alpha = a + b + 2c$. By this, $(p, q) = (a + c, b + c)$-forms map to $(a + (n - a - b - c), b + (n - a - b - c)) = (n - q, n - p)$-forms as desired. \qed
Let $L : \Omega^{p,q} \to \Omega^{p+1,q+1}$ be the map $\alpha \mapsto \omega \wedge \alpha$, $L^* : \Omega^{p,q} \to \Omega^{p-1,q-1}$ the adjoint map $\alpha \mapsto (-1)^{p+q} L^* \alpha$. Furthermore, set $d_C = J^{-1} dJ = (-1)^{k+1} J dJ$, with adjoint $d_C^* = J^{-1} d^* J = (-1)^{k+1} J d^* J$. On functions, we have that
\begin{equation}
  d_c f = -J df = -J (\partial f + \bar{\partial} f) = -i(\partial - \bar{\partial}) f
\end{equation}
which extends to higher forms as well. Thus, $dd_C = -i(\partial + \bar{\partial}) (\partial - \bar{\partial}) = 2i\bar{\partial}\partial f$.

**Lemma 1.** For $X$ Kähler, $[L, d] = 0, [L^*, d^*] = 0$, $[L, d^*] = d_C, [L^*, d] = -d_C^*$.

**Proof.** The first part follows from $d(\alpha \wedge \omega) = \alpha \wedge \omega$. For the second, see Wells, theorem 4.8.

**Proposition 2.** $\Delta_C = J^{-1} \Delta J = d_C d_C^* + d_C^* d_C = \Delta$

**Proof.** By $J$-invariance of $\omega$, we have that $[L, J] = [L^*, J] = 0$. Using the above identities, we have that $[L^*, d_C] = d^*$, so
\begin{equation}
  \Delta = dd^* + d^* d = d[L^*, d_C] + [L^*, d_C] d = dL^* d_C - dd_C L^* + L^* d_C d - d_C L^* d
\end{equation}
Conjugating by $J$ simply swaps terms, since $dd_C = -d_C d$.

Let
\begin{equation}
  \bar{\partial}^* = - \partial * : \Omega^{p,q} \to \Omega^{p,q-1}
\end{equation}
\begin{equation}
  \partial^* = - \bar{\partial} * : \Omega^{p,q} \to \Omega^{p-1,q}
\end{equation}
so $d^* = \partial^* + \bar{\partial}^*$.

**Lemma 2.** $\partial^*$ is $L^2$-adjoint to $\bar{\partial}$, and $\bar{\partial}^*$ is $L^2$-adjoint to $\partial$.

For $\phi, \psi \in \Omega^k(M, \mathbb{C})$, we have the natural scalar product
\begin{equation}
  \langle \phi, \psi \rangle_{L^2} = \int_M \phi \wedge *\bar{\psi}
\end{equation}
Under this, the various $\Omega^{p,q}$ are orthogonal because if $\phi \in \Omega^{p,q}, \psi \in \Omega^{p',q'}, (p,q) \neq (p',q')$, then $\phi \wedge *\bar{\psi}$ is of type
\begin{equation}
  (n + (p - p'), n + (q - q')) \neq (n, n)
\end{equation}
Finally, define the operators
\begin{equation}
  \square = \partial \partial^* + \partial^* \partial, \square = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial} : \Omega^{p,q} \to \Omega^{p,q}
\end{equation}

**Theorem 6.** For $M$ compact, Kähler,
\begin{equation}
  H^k_{\square}(M) = \mathcal{H}_{\square}^{p,q} = \ker \square
\end{equation}

The proof that each $\square$-cohomology class contains a unique $\square$-harmonic form is similar to that of the Hodge theorem in the Riemannian case.

**Theorem 7.** $\Delta = 2 \square = 2 \bar{\square}$.

**Proof.** By the first lemma, $d^* d_c = d^* [L, d^*] = d^* L d^* = -[L, d^*] d^* = -d_C d^*$. Moreover, $d_c = -i(\partial - \bar{\partial})$, so $\bar{\partial} = \frac{1}{2} (d - id_c)$ and $\bar{\partial}^* = \frac{1}{2} (d^* + id_c^*)$. Thus,
\begin{equation}
  4 \square = (d - id_c) (d^* + id_c^*) + (d^* + id_c^*) (d - id_c)
\end{equation}
\begin{equation}
  = (dd^* + d^* d) + (d_c d_c^* + d_c^* d_c) + i(dd_c^* + d_c^* d) - i(d_c d_c^* + d_c^* d_c)
\end{equation}
\begin{equation}
  = \Delta + \Delta_c + 0 + 0 = 2 \Delta
\end{equation}

**Corollary 4.** $\Delta$ maps $\Omega^{p,q}$ to itself and
\begin{equation}
  H^k_{dR}(M, \mathbb{C}) = \mathcal{H}_\Delta^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q} = \bigoplus_{p-q} H^p_{\square}(M)
\end{equation}