SYMPLECTIC GEOMETRY, LECTURE 17

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The Hodge decomposition stated last time places strong constraints on $H^*$ of Kähler manifolds, e.g. dim $H^k$ is even for $k$ odd because C conjugation gives isomorphisms $\overline{H^{k,q}} \cong H^{q,k}$ (note that this is false for symplectic manifolds in general). The Hodge star * gives isomorphisms $H^{p,q} \cong H^{n-q,p}$ and the Hodge diamond structure on the the ranks of the Dolbeault cohomology groups, i.e.

\[
\begin{array}{cccc}
  h^{n,n} & \cdots & \cdots & h^{0,n} \\
  \vdots & \ddots & \ddots & \vdots \\
  h^{n,0} & \cdots & h^{1,0} & h^{0,0}
\end{array}
\]

(1)

is symmetric across the two diagonal axes. Moreover, note that $[\omega^\wedge p] \in H^{p,p}$ is nonzero, since $[\omega^\wedge n]$ is the volume class.

We have even stronger constraints, namely the “hard Lefschetz theorem”.

**Theorem 1.** $L^{n-k} = (\cdot \wedge \omega^{n-k}) : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$ is an isomorphism.

This is false for many symplectic manifolds. Moreover, combining this with Poincaré duality gives that, for $k \leq n$, $H^k \times H^k \to \mathbb{R}$, $\alpha, \beta \mapsto \int \alpha \cap \beta \wedge \omega^{n-k}$ is a nondegenerate bilinear pairing (skew-symmetric if $k$ is odd).

We also have the Kodaira embedding theorem:

**Theorem 2.** For $(X, \omega)$ a compact Kähler manifold, $[\omega] \in H^2(X, \mathbb{Z})$, $\exists$ a projective embedding $X \to \mathbb{C}P^N$ realizing $X$ as a projective algebraic variety.

We will see a symplectic geometry proof due to Donaldson.

1. HOLONOMIC VECTOR BUNDLES

Let $(M, J)$ be a complex manifold, $E \to M$ a complex vector bundle. Then we can cover $M$ by $U_\alpha$ s.t. the restrictions $U_\alpha \times \mathbb{C}^n \cong E|_{U_\alpha} \to U_\alpha$ are trivial.

**Definition 1.** $E$ is a holomorphic vector bundle if the transition functions $\phi_{\alpha, \beta} : U_\alpha \cap U_\beta \to \text{GL}(r, \mathbb{C})$ are holomorphic.

Note that this only makes sense on a complex manifold. Now, $\exists$ a natural $\overline{\partial}$ operator on sections given in a local trivialization by $\overline{\partial}$ (given a section $s$ which looks like $\zeta_s$ in the local trivialization $\alpha$, on an intersection we have that $\overline{\partial}\zeta_s = \phi_{\alpha, \beta} \overline{\partial}\zeta_{s'}$ since $\overline{\partial}\phi_{\alpha, \beta} = 0$). This extends to $\overline{\partial} : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ similarly.

**Definition 2.** $\overline{\partial}(E) = \text{Ker}(\overline{\partial}: \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)) / \text{Im}(\overline{\partial}: \Omega^{p,q+1}(E) \to \Omega^{p,q+2}(E))$. In particular, $\overline{\partial}(E)$ is the space of holomorphic sections.

Specifying the holomorphic structure on a complex vector bundle $E$ is equivalent to specifying a $\overline{\partial}$ operator with $\overline{\partial}^2 = 0$. The $\overline{\partial}$ operator is half of a connection: in fact, $\nabla$ a connection on $E$ decomposes into $\nabla = \nabla^{1,0} + \nabla^{0,1}$.

**Proposition 1.** For $(E, \overline{\partial}, \cdot|\cdot)$ a holomorphic bundle with a Hermitian metric, $\exists!$ Hermitian connection s.t. $\nabla^{0,1} = \overline{\partial}$.

**Proof.** We work in local coordinates on $M$, and local trivializations of $E$ by orthonormal sections $\sigma_j$ (but not necessarily holomorphic trivializations; $\overline{\partial}\sigma_j$ may be nonzero). $\nabla = d + A$ for $A = (a_{ij})$ a matrix-valued 1-form ($a_{ij} = \langle \overline{\partial}\sigma_j, \sigma_i \rangle$). $\nabla$ is Hermitian iff $a_{ij} = -\overline{a}_{ij}$, i.e. $A$ is antihermitian, and $\nabla$ is holomorphic, i.e. $\nabla^{0,1} = \overline{\partial}s$ iff $A^{0,1}$ is given by $a^{0,1}_{ij} = \overline{\partial}s_{ij}$. Then $A^* = -A \Leftrightarrow A^{1,0} = -(A^{0,1})^*$, i.e. $a^{1,0}_{ij} = -a^{0,1}_{ij}$. \qed
Equivalently, in a holomorphic trivialization, when $\bar{\partial}$ is the usual $\partial$ operator, $\langle \cdot, \cdot \rangle$ given by $h = C^\infty$ function with values in positive definite Hermitian matrices, $\nabla = d + A$ again and $\nabla$ is Hermitian $\iff d\langle s, s' \rangle = \langle \nabla s, s' \rangle + \langle s, \nabla s' \rangle \iff d(s^*hs') = (ds^* + s^*A^*)hs' + s^*h(ds' + As') \iff dh = A^*h + hA$. On the other hand, now $\nabla^{0,1} = \bar{\partial} \iff A^{1,0} = 0$. Hence $dh = A^*h + hA \iff A = h^{-1}\partial h$ (and $A^* = \bar{\partial}h \cdot h^{-1}$).

**Proposition 2.** In a holomorphic frame, the connection 1-form $A$ is of type $(1,0)$, and $\partial A = -A \wedge A$, $R^\nabla = \bar{\partial}A$ is of type $(1,1)$, and $\partial R = 0$ and $\partial R = [R, A]$.

In fact, we have

**Theorem 3.** $(E, \nabla^{0,1} = \bar{\partial}^\nabla)$ is holomorphic $\iff (\bar{\partial}^\nabla)^2 = 0 \iff R^{0,2} = 0$.

**Proof.** First, $A = h^{-1}\partial h$ has type $(1,0)$ by the above, and

\begin{equation}
\partial A = \partial(h^{-1}) \wedge \partial h = -(h^{-1}(\partial h)h^{-1}) \wedge \partial h = -h^{-1}(\partial h) \wedge (h^{-1}\partial h) = -A \wedge A
\end{equation}

by the formula for derivatives of inverses in a noncommutative setting. Second, $R^\nabla = dA + A \wedge A = dA - \partial A = \bar{\partial}A$, hence it has type $(1,1)$. Finally, $\partial R = \bar{\partial}\partial A = 0, \partial \partial R = \partial \partial A = -\bar{\partial}\partial A = \bar{\partial}A \wedge A - A \wedge \bar{\partial}A = [R, A]$. \qed