SYMPLECTIC GEOMETRY, LECTURE 19

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We now return to the complex Kähler case. Let $(M, \omega, J)$ be a complex Kähler manifold.

**Proposition 1** (Donaldson). $\exists$ a family of sections $(\sigma_{k,p})_{k>p} \in M$ which is uniformly bounded and almost-holomorphic, uniformly concentrated, and satisfies $|\sigma_{k,p}| \geq c > 0$ on $B(p, k^{-1/2})$. Furthermore, $\exists$ a family of holomorphic sections $(\tilde{\sigma}_{k,p})$ with $\sup |\sigma_{k,p} - \tilde{\sigma}_{k,p}|, \sup (k^{1/2} |\nabla \sigma_{k,p} - \nabla \tilde{\sigma}_{k,p}|) \leq O(\exp(-\lambda k^{1/3}))$. That is, the $\tilde{\sigma}_{k,p}$ are so close to $\sigma_{k,p}$ that they’re interchangeable in practice.

**Proof.** Fix $p \in M$ and holomorphic coordinates $(M, p) \to (\mathbb{C}^n, 0)$ (not necessarily Darboux). We can choose the coordinates to be isometric at the origin.

1. Let $u$ be a local section of $L$ near $p$ which is holomorphic and s.t. $|u(x)| = 1$ (e.g. $u \equiv 1$ in a holomorphic trivialization). Then

   \[
   \partial \bar{\partial} \log |u|^2 = \partial \bar{\partial} (u^{-1} \partial \bar{\partial} u) = u^{-1} \partial \bar{\partial} \bar{\partial} u = R^{1,1} = -i\omega
   \]

   with the third equality coming from $(R^{1,1}) = \partial \bar{\partial} \partial \bar{\partial} + \partial \bar{\partial} \partial \bar{\partial} = (\partial \bar{\partial})^{1,1}$ and $\partial \bar{\partial} u = 0$. In local coordinates, we can write

   \[
   \log |u|^2 = \sum_j (f_j z_j + \bar{f}_j \bar{z}_j) + \sum_{ij} \sum_j (g_{ij} z_i z_j + h_{ij} z_i z_j + f_{ij} z_i z_j) + O(|z|^3)
   \]

   Replacing $u$ by $\exp(\sum -f_j z_j - h_{ij} z_i z_j)u$ (which preserves holomorphicity), we can assume $\log |u|^2 = \sum g_{ij} z_i z_j + O(|z|^3)$. $\partial \partial \log |u|^2 = -i\omega \implies (g_{ij}) = -\frac{1}{2} (\text{metric tensor on } T_M) \implies \log |u|^2 = -\frac{1}{2} |z|^2 + O(|z|^3).$ Hence $u^k$ is a local holomorphic section of $L^\otimes k$, $|u|^2 = \exp(-\frac{k}{3} |z|^2 + kO(|z|^3))$. Estimating the growth of derivatives of $\log |u|^2$ gives us uniform concentratedness estimates as long as $|z| < 1$ (which is fine since the “support” of $u^k$ a ball of radius $\frac{1}{\sqrt{k}}$). Then let $\sigma_{k,p}(q) = \chi_k(\text{dist}(p, q))u(q)^k$, where $\chi_k$ is a smooth cut-off function at distance $\sim k^{-1/3}$ (i.e. $\chi_k \equiv 1$ inside the ball of radius $k^{-1/3}$ and 0 outside a larger ball).

   Note that the cutoff occurs in the region where $|z| \sim k^{-1/3}$ i.e. $|u^k| \sim \exp(-k |z|^2) \sim \exp(-1/3).$ Thus we get $\sup |\partial \nabla \sigma_{k,p}| = \sup |u^k \partial \nabla \chi_k| \leq O(\exp(-\lambda k^{1/3}))$ since $\partial \chi_k \equiv 0$ except for $|z| \sim k^{-1/3}$ and $|\partial \chi_k| \leq k^{1/3}.$

2. To obtain the $\tilde{\sigma}_{k,p}$, we use the following lemma:

   **Lemma 1.** $\forall s \in \Gamma(L^\otimes k), \exists \xi \in \Gamma(L^\otimes k)$ s.t. $\|\xi\|_{L^2} \leq \frac{c}{\sqrt{k}} \|\partial \nabla s\|_{L^2}$ and $s + \xi$ is holomorphic.

   We apply this lemma to $\sigma_{k,p}$ and obtain $\|\xi\|_{L^2} \leq \frac{c}{\sqrt{k}} \|\partial \nabla \sigma_{k,p}\|_{L^2} \leq O(k^{-2n/3-1/2}\exp(-\lambda k^{-1/3}))$, where the $L^2$ estimate on $\partial \nabla \sigma_{k,p}$ follows from the pointwise bound and the observation that it is supported in a ball of volume $\sim k^{-2n/3}$. To get a pointwise $C^r$-estimate on $\xi$, we use a Cauchy estimate expressing values of holomorphic functions at $q$ by integrals over balls containing $q$. At points inside $B(p, k^{-1/3})$, $\chi = 1$ so $\sigma_{k,p}$ is holomorphic there, as is $\xi$, and $\|\xi\|_{C^r}$ is controlled by $\|\xi\|_{L^2} \sim \exp(-\lambda k^{1/3})$ on $B(k^{-1/3})$. Finally, the Cauchy estimates for $\sigma_{k,p} + \xi$ imply that $\|\sigma_{k,p} + \xi\|_{C^r}$ is also controlled by the local $L^2$ norm and thus also bounded by $\exp(-\lambda k^{1/3})$ outside of $B(p, k^{-1/3})$ as desired.

\[\square\]
Injectivity:

\[ \partial \alpha = \sum_i e^i \wedge \nabla \tau \alpha \]

\[ \partial^* \alpha = - \sum_i g(e^i,e^j) i_{\tau}(\nabla e_i, \alpha) \]

Take a frame that’s orthonormal at the origin, and radially parallel transport so \( \nabla e_i, e_j = 0 \) at the origin; this preserves type \((1,0)\) forms since \( J \) is integrable. Then

\[ \Delta_k \alpha = - \sum_{ij} (e_j \wedge \nabla e_i, \nabla \tau \alpha) - \sum_i e^j \wedge (i_{\tau} \nabla \tau \nabla e_i, \alpha) \]

\[ = \sum_i -\nabla e_i, \nabla \tau \alpha + \sum_{ij} e^j \wedge (i_{\tau} R^{TM} \Omega^{k}(e_i, \tau_k) \alpha) \]

\[ = D\alpha + R\alpha + k\alpha \]

because at the origin \( R^{k}(e_i, \tau_j) = -ik\omega(e_i, \tau_j) = k\delta_{ij} \). \( D \) is semipositive, since \( \langle D\alpha, \alpha \rangle = \sum ||\nabla \tau \alpha||^2 + d(\text{something}) \Rightarrow \int_M \langle D\alpha, \alpha \rangle \geq 0 \). Therefore, for \( k \) large enough, \( \Delta_k \) is invertible and \( \exists \) an inverse \( G \) of norm \( O(\frac{1}{k}) \).

Given \( s \in \Gamma(L^k) \), set \( \xi = -\partial \tau G\tilde{s} \). Then

\( (s + \xi) \) is holomorphic since

\[ \tilde{\partial}(s + \xi) = \tilde{\partial}s - \tilde{\partial \tau} G\tilde{s} = \tilde{s} - (\Delta_k - \tilde{\partial} \tilde{\partial})G\tilde{s} = \tilde{\partial} \tilde{\partial} G\tilde{s} \]

\[ \text{but } \text{Im}\tilde{\partial} \cap \text{Im}\tilde{\partial}^* = 0 \text{ by Hodge theory, so } \tilde{\partial}(s + \xi) = 0. \]

\[ ||\xi||^2 = \langle \tilde{\partial} \tilde{\partial} G\tilde{s}, \tilde{\partial} \tilde{\partial} G\tilde{s} \rangle = \langle \tilde{\partial} \tilde{\partial} G\tilde{s}, G\tilde{s} \rangle = \langle \tilde{s}, G\tilde{s} \rangle \leq ||G|| ||\tilde{s}||^2 \leq kc^{-1} ||\tilde{s}||^2 \]

This completes the proof. \( \square \)

Going from these collections of sections to the Kodaira embedding is straightforward:

- Well-definedness: we need that \( \forall p, \exists s \in H^0(L^k) \) s.t. \( s(p) \neq 0 \), which comes from the fact that \( |\tilde{s}_{k,p}(p)| \simeq 1 \neq 0. \)
- Immersion: need that \( \forall p \in M, v \in T_p M, \exists \sigma_1, \sigma_2 \in H^0(L^k) \) s.t. \( d_v(\tilde{s}_{k,p}) \neq 0 \). This would give us a projection to a certain \( \mathbb{CP}^1 \) factor of \( \mathbb{CP}^n \) which has nonzero derivative in the direction of \( v \). We could do this by looking at \( \tilde{s}_{k,\tau^\pm}, q_{\pm} = \exp_p(\pm k^{-1/2} v) \). More simply, we set \( \sigma_2 = \tilde{s}_{k,p}, \sigma_1 \) and adding \( \xi \) to make \( \xi \) a nonzero derivative. Then \( \tilde{s}_{k,p} = z_1 + \cdots \Rightarrow d_v(\tilde{s}_{k,p}) \neq 0. \)
- Injectivity: If \( p, q \) are at a distance \( << k^{-1/3} \) then (using the above argument for immerseness) the sections are different at \( p \) and \( q \). If the distance is greater, \( |\tilde{s}_{k,p} : \tilde{s}_{k,q}| \simeq |1 : 0| \) and \( |0 : 1| \) respectively.